DIFFERENTIAL EQUATIONS IN THE AP* CALCULUS EXAM

Victor Liu

Olympia Press   San Francisco   California

All inquiries should be addressed to:
Olympia Press
950 Clement Street
San Francisco, CA 94118

International Standard Book No. 0-9727892-1-9

Contents

* These chapters cover material beyond the scope of the AP exam.

Preface

After taking the AP Calculus BC exam in 2001, I noticed that many of the self-study books did not provide sufficient material on differential equations in the test, especially on the logistic equation and its applications. So I wrote this book mainly for helping students prepare for the differential equations covered in the AP Calculus exam. Specifically, the book covers the following topics: separable differential equations, with emphasis on the exponential and logistic growth and their applications, Euler's method for solving differential equations numerically, and slope fields for visualizing differential equations. Among these topics, the logistic equation, Euler's method and slope fields are only covered in the AP Calculus BC exam, but slope fields will be included in the 2004 AB exam.

The book sometimes goes beyond what is normally covered in the AP exam. The main reasons for this are that the added material can help students better understand the topics on the test, and that many calculus classes cover some of the other topics. Some topics, such as the implicit forms of the differential equations and the hyperbolic forms, are meant to enhance the student's interest and expand his or her general knowledge of mathematics.

This book is not meant to be a classroom text, but rather a supplement to self-study books. This book will take you through all the material in a textbook manner and then you will be able to practice for the real test with three sample free response problems (with answers) in the back of the book.

Acknowledgments

I am very grateful to Professor Kuo Chen, principal of Olympia Institute, for his guidance and encouragement. He has been a great inspiration, and has contributed greatly to this effort by providing materials and research, not to mention countless hours of editing and invaluable advice. Without Professor Chen, this book would not have been possible.

I thank Mr. DeRuiter, my AP Calculus teacher at Monta Vista High School, for showing me how interesting calculus can be.

I would like to thank Dr. Sandra Song for imparting upon me some of her vast knowledge of chemistry and helping me find examples in this area.

I would also like to thank my dad, Ralph, for his continual support, suggestions, and countless hours of proofreading.

And finally, thanks to Dr. Donald Knuth for creating the TeX typesetting system, which saved me countless hours of formatting time while writing this book.

Chapter 1

Separable Differential Equations

Purpose: *To learn how to solve separable differential equations. This is required by the AP Calculus AB/BC exams.*

A **differential equation** is an equation that involves derivatives of a function. A **solution** to a differential equation is any function that can satisfy it. For example, the solution to the differential equation $\frac{dy}{dx} + 4y = 0$ is $y = Ce^{-4x}$ because if the solution is substituted into the original equation, the result is a true statement:

$$\frac{d}{dx}Ce^{-4x} + 4Ce^{-4x} = 0$$

$$-4Ce^{-4x} + 4Ce^{-4x} = 0$$

$$0 = 0$$

Notice that there is an arbitrary constant C in the solution. This is a result of applying indefinite integration in the process of finding the solution. Since C can be any constant in the example, this form is called the **general solution** of the differential equation. If some **initial condition** is given, such as $y(1) = 2$ in the above example, then a unique value of the constant C can be determined by substituting the initial condition into the general solution:

$$2 = Ce^{-4(1)}$$

$$C = 2e^4$$

In this case the differential equation has a **particular solution**:

$$y = Ce^{-4x} = 2e^4 \cdot e^{-4x} = 2e^{4(1-x)}$$

A **separable differential equation** is a differential equation that can be written in the following form:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)} \text{ or } f(x)\,dx = g(y)\,dy$$

(Separable differential equation forms)

In the above form with variables x and y separated on each side of the equation, the solution to the differential equation can be found by integrating both sides of the equation:

$$\int g(y)\,dy = \int f(x)\,dx$$

Sometimes a differential equation is not directly separable, but can be converted to a separable equation by some mathematical manipulations. We will discuss this case in the next chapter.

Example 1.1
Find the general solution of the differential equation $y' - y\sin x = 0$.

Solution:
The above differential equation can be written as:

$$\frac{dy}{dx} = y\sin x$$

$$\frac{dy}{y} = \sin x dx$$

Now by integrating both sides we have a general solution:

$$\int \frac{dy}{y} = \int \sin x dx$$

$$\ln|y| = -\cos x + C$$

$$|y| = C_1 e^{-\cos x} \quad (C_1 = e^C)$$

To verify the above solution, substitute $y = \pm C_1 e^{-\cos x}$ into the original equation.
If $y = +C_1 e^{-\cos x}$,

$$y' - y\sin x = C_1 e^{-\cos x}\sin x - C_1 e^{-\cos x}\sin x = 0$$

If $y = -C_1 e^{-\cos x}$,

$$y' - y\sin x = -C_1 e^{-\cos x}\sin x + C_1 e^{-\cos x}\sin x = 0$$

Since both substitutions resulted in true statements, $|y| = C_1 e^{-\cos x}$ is the general solution.

► TIP

While changing the constant of integration for convenience, use a different expression, for example, use C_1 instead of C in the above case, so that their meanings are consistent.

Example 1.2
Find the particular solution to the differential equation $\frac{dy}{dx} - \sqrt{y} = x\sqrt{y}$, if $y = 9$ when $x = 4$.

Solution:
Separate the variables first:

$$\frac{dy}{dx} = x\sqrt{y} + \sqrt{y}$$

$$\frac{dy}{dx} = (x+1)\sqrt{y}$$

$$\frac{dy}{\sqrt{y}} = (x+1)\,dx$$

$$\int y^{-\frac{1}{2}} dy = \int (x+1)\, dx$$

$$2y^{\frac{1}{2}} = \frac{1}{2}x^2 + x + C$$

$$y = \left(\frac{1}{4}x^2 + \frac{1}{2}x + C_1\right)^2 \quad (C_1 = \tfrac{1}{2}C)$$

Now substituting the initial condition:

$$9 = \left(\frac{1}{4}(4)^2 + \frac{1}{2}(4) + C_1\right)^2$$

$$\pm 3 = 4 + 2 + C_1$$

$$C_1 = -6 \pm 3 = -9, -3$$

So the particular solutions to the differential equation are:

$$y = \left(\frac{1}{4}x^2 + \frac{1}{2}x - 9\right)^2 \text{ and } y = \left(\frac{1}{4}x^2 + \frac{1}{2}x - 3\right)^2$$

Verify the solutions by substituting them into the original equation. If $y = \left(\frac{1}{4}x^2 + \frac{1}{2}x - 9\right)^2$,

$$\frac{dy}{dx} = 2\left(\frac{1}{4}x^2 + \frac{1}{2}x - 9\right)\left(\frac{1}{2}x + \frac{1}{2}\right) = \left(\frac{1}{4}x^2 + \frac{1}{2}x - 9\right)(x+1) = \sqrt{y}\,(x+1)$$

When $x = 4$,

$$y = \left(\frac{1}{4}(4)^2 + \frac{1}{2}(4) - 9\right)^2 = (-3)^2 = 9$$

The other solution $y = \left(\frac{1}{4}x^2 + \frac{1}{2}x - 3\right)^2$ can be verified similarly.

► NOTE

Sometimes it is possible to obtain more than one particular solution if square roots are involved. If the solutions are applied to a real world situation, be sure to check whether both solutions make sense.

Example 1.3

Find the particular solution to the differential equation $\frac{dx}{dt} = r(a-x)^2$, with the initial condition $x(0) = x_0$.

Solution:

Separating variables x and t, and integrating both sides of the equation, we have

$$\int \frac{dx}{(a-x)^2} = \int r\,dt$$

Letting $u = a - x$, then $dx = -du$, substitute these into the above equation,

$$\int \frac{-du}{u^2} = \int r dt$$

$$\frac{1}{u} = rt + C$$

$$\frac{1}{a-x} = rt + C$$

Apply the initial condition $x(0) = x_0$ to the above equation to get $C = \frac{1}{a-x_0}$. So the particular solution can be obtained:

$$\frac{1}{a-x} = rt + \frac{1}{a-x_0} = \frac{r(a-x_0)t + 1}{a-x_0}$$

$$a - x = \frac{a-x_0}{1 + r(a-x_0)t}$$

$$x = a - \frac{a-x_0}{1 + r(a-x_0)t}$$

Example 1.4

Find the particular solution to the differential equation $\frac{dy}{dx} = (a-y)(b-y)$, $b > a > 0$, $y \neq a$, $y \neq b$, with the initial condition $y(0) = y_0$.

Solution:

Separating the variables, we have $\frac{dy}{(a-y)(b-y)} = dx$. Performing a partial fraction decomposition, we obtain

$$\frac{1}{(a-y)(b-y)} = \frac{1}{(b-a)(a-y)} + \frac{1}{(a-b)(b-y)}$$

So the original equation becomes

$$\int \frac{dy}{(b-a)(a-y)} + \int \frac{dy}{(a-b)(b-y)} = \int dx$$

$$\frac{1}{b-a}(-\ln|a-y|) + \frac{1}{a-b}(-\ln|b-y|) = x + C$$

$$\ln|b-y| - \ln|a-y| = (b-a)x + C_1 \quad (C_1 = (b-a)C)$$

$$\ln\left|\frac{b-y}{a-y}\right| = (b-a)x + C_1$$

If $a < y < b$,

$$b - y = (y-a)C_2 e^{(b-a)x} \quad (C_2 = e^{C_1})$$

$$y + yC_2 e^{(b-a)x} = b + aC_2 e^{(b-a)x}$$

$$y = \frac{b + aC_2 e^{(b-a)x}}{1 + C_2 e^{(b-a)x}}$$

$$y = \frac{(b-a) + \left(a + aC_2 e^{(b-a)x}\right)}{1 + C_2 e^{(b-a)x}}$$

$$y = a + \frac{b-a}{1 + C_2 e^{(b-a)x}}$$

Apply the initial condition $y(0) = y_0$,

$$y_0 = a + \frac{b-a}{1 + C_2}$$

$$1 + C_2 = \frac{b-a}{y_0 - a}$$

$$C_2 = \frac{b - a - y_0 + a}{y_0 - a} = \frac{b - y_0}{y_0 - a}$$

Substitute C_2 back in the y expression,

$$y = a + \frac{b-a}{1 + \left(\frac{b-y_0}{y_0-a}\right) e^{(b-a)x}}$$

In the case of $y < a$ or $y > b$,

$$b - y = (a - y) C_2 e^{(b-a)x}$$

$$y - yC_2 e^{(b-a)x} = b - aC_2 e^{(b-a)x}$$

$$y = \frac{b - aC_2 e^{(b-a)x}}{1 - C_2 e^{(b-a)x}}$$

$$y = \frac{(b-a) + \left(a - aC_2 e^{(b-a)x}\right)}{1 - C_2 e^{(b-a)x}}$$

$$y = a + \frac{b-a}{1 - C_2 e^{(b-a)x}}$$

Apply the initial condition again,

$$y_0 = a + \frac{b-a}{1 - C_2}$$

$$1 - C_2 = \frac{b-a}{y_0 - a}$$

$$C_2 = \frac{y_0 - a - b + a}{y_0 - a} = \frac{y_0 - b}{y_0 - a}$$

Substitute C_2 in the y expression,

$$y = a + \frac{b-a}{1 - \left(\frac{y_0-b}{y_0-a}\right) e^{(b-a)x}} = a + \frac{b-a}{1 + \left(\frac{b-y_0}{y_0-a}\right) e^{(b-a)x}}$$

We have obtained the same specific solution in both cases. To verify the solution, substitute $y = a + \frac{b-a}{1+\left(\frac{b-y_0}{y_0-a}\right)e^{(b-a)x}}$ into the original equation $\frac{dy}{dx} = (a-y)(b-y)$:

$$\frac{dy}{dx} = \frac{-(b-a)\left[\left(\frac{b-y_0}{y_0-a}\right)e^{(b-a)x}(b-a)\right]}{\left[1+\left(\frac{b-y_0}{y_0-a}\right)e^{(b-a)x}\right]^2} = \frac{-(b-a)^2\left(\frac{b-y_0}{y_0-a}\right)e^{(b-a)x}}{\left[1+\left(\frac{b-y_0}{y_0-a}\right)e^{(b-a)x}\right]^2}$$

$$(a-y)(b-y) = -\frac{b-a}{1+\left(\frac{b-y_0}{y_0-a}\right)e^{(b-a)x}}\left[b-a-\frac{b-a}{1+\left(\frac{b-y_0}{y_0-a}\right)e^{(b-a)x}}\right] = \frac{-(b-a)^2\left(\frac{b-y_0}{y_0-a}\right)e^{(b-a)x}}{\left[1+\left(\frac{b-y_0}{y_0-a}\right)e^{(b-a)x}\right]^2}$$

► NOTE

When removing the absolute value sign in an expression, such as $\left|\frac{b-y}{a-y}\right|$ in the above example, it is necessary to discuss two cases (unless there is only one possibility), one is to assume its argument is positive and the other is negative.

Practice problem set 1

Solve the following separable differential equations:

1. $2xydx - (1+x^2)\,dy = 0$
2. $\sqrt{xy}\frac{dy}{dx} = 3$
3. $x^2(y-1)\,dx - y^2(x+1)\,dy = 0$
4. $dx + (1-x^2)\cot y dy = 0$
5. $\frac{3}{t}dt - \frac{y-2}{y}dy = 0$
6. $\cos xdx + 2ydy = 0$; $y(0) = 1$
7. $y' = \frac{xy-2y}{y^2+1}$; $y(2) = 1$
8. $xe^{x^2}dx + (y^3-1)\,dy = 0$; $y(0) = 2$
9. $\frac{dy}{dx} = \frac{3x-1}{(x-3)(x+1)}$; $y(0) = 0$
10. $\frac{dy}{dx} = y(a-y)$; $a > 0$, $y \neq a$, $y \neq 0$

Chapter 2

Reduction to Separable Equations*

Purpose: *To learn how to convert several types of differential equations into separable equations and solve them. The material in this chapter is not covered on the AP Calculus exam.*

Separation of variables is one of the basic techniques for solving differential equations. In this chapter we are going to learn several types of differential equations that are not directly separable, but can be reduced to separable equations by simple mathematical manipulations. Although the content of this chapter is not a requirement of the AP Calculus exam, you are encouraged to read this chapter to enhance your skills of solving differential equations.

Homogeneous Equations

Homogeneous differential equations in the form of $\frac{dy}{dx} = f(x,y)$ have the property that $f(tx,ty) = f(x,y)$. For example, in the equation $\frac{dy}{dx} = \frac{x+y}{2y}$, $f(x,y) = \frac{x+y}{2y}$. Since $f(tx,ty) = \frac{tx+ty}{2ty} = f(x,y)$, the equation is homogeneous. A homogeneous equation can be transformed into a separable equation by making the substitution: $y = vx$, where v is a function of x. Thus,

$$\frac{dy}{dx} = x\frac{dv}{dx} + v$$

► TIP

A simple way to check whether an equation is homogeneous is to make sure that all the terms in $f(x,y)$ have the same degree.

Example 2.1

Solve the differential equation $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$.

Solution:

Since $f(tx,ty) = \frac{t^2x^2+t^2y^2}{2txty} = f(x,y)$, the equation is homogeneous (notice that all the terms in $\frac{x^2+y^2}{2xy}$ have degree 2). Make the substitution $y = vx$ then $v = \frac{y}{x}$, and $\frac{dy}{dx} = v + x\frac{dv}{dx}$. So the original equation becomes:

$$v + x\frac{dv}{dx} = \frac{x^2 + x^2v^2}{2x^2v}$$

$$x\frac{dv}{dx} = \frac{1+v^2}{2v} - v = \frac{1-v^2}{2v}$$

The above equation can be solved by separating the variables v and x and integrating both sides:

$$\int \frac{2v\,dv}{1-v^2} = \int \frac{dx}{x}$$

$$-\ln\left|1-v^2\right| = \ln|x| + C$$

$$\left|1-v^2\right| = \frac{C_1}{|x|} \quad (C_1 = e^{-C})$$

To get rid of the absolute value signs on both sides of the equation, we need to assume there are two cases: $1-v^2 = \frac{C_1}{x}$ and $1-v^2 = -\frac{C_1}{x}$, therefore

$$1-v^2 = \pm\frac{C_1}{x}$$

Substitute $v = \frac{y}{x}$ into the above equation:

$$1-\frac{y^2}{x^2} = \pm\frac{C_1}{x}$$

$$y^2 = x^2 \mp C_1 x$$

To verify the solution, differentiate both sides of it with respect to x:

$$2y\frac{dy}{dx} = 2x \mp C_1$$

Therefore $\frac{dy}{dx} = \frac{2x \mp C_1}{2y}$ and $\frac{x^2+y^2}{2xy} = \frac{x^2+x^2\mp C_1 x}{2xy} = \frac{2x\mp C_1}{2y} = \frac{dy}{dx}$

Linear Fractional Equations

A linear fractional equation has the form $\frac{dy}{dx} = \frac{a_1x+b_1x+c_1}{a_2x+b_2x+c_2}$, where a_1, b_1, a_2, and b_2 are non-zero constants. A special case of the equation is when $\frac{a_1}{a_2} = \frac{b_1}{b_2} = k$. Under this condition, linear fractional equations can be reduced to separable equations by making the substitution $v = a_1x + b_2y$. Since $a_2 = \frac{a_1}{k}$ and $b_2 = \frac{b_1}{k}$, we have $a_2x + b_2x = \frac{1}{k}(a_1x + b_1x) = \frac{v}{k}$, and also $\frac{dv}{dx} = a_1 + b_1\frac{dy}{dx}$ or $\frac{dy}{dx} = \frac{1}{b_1}\left(\frac{dv}{dx} - a_1\right)$.

Example 2.2

Solve the differential equation $\frac{dy}{dx} = \frac{2x+3y+5}{4x+6y-3}$.

Solution:

Make the substitution $v = 2x+3y$, then $\frac{dv}{dx} = 2 + 3\frac{dy}{dx}$, or $\frac{dy}{dx} = \frac{1}{3}\left(\frac{dv}{dx} - 2\right)$. Substitute these into the original equation:

$$\frac{1}{3}\left(\frac{dv}{dx} - 2\right) = \frac{v+5}{2v-3}$$

$$\frac{dv}{dx} = \frac{3(v+5)}{2v-3} + 2 = \frac{7v+9}{2v-3}$$

$$\int \frac{2v-3}{7v+9}dv = \int dx$$

Since $\frac{2v-3}{7v+9} = \frac{7(2v-3)}{7(7v+9)} = \frac{14v-21+(18-18)}{7(7v+9)} = \frac{(14v+18)-(21+18)}{7(7v+9)} = \frac{2}{7} - \frac{39}{7(7v+9)}$,

$$\int \left(\frac{2}{7} - \frac{39}{7(7v+9)}\right) dv = \int dx$$

$$\frac{2}{7}v - \frac{39}{49}\ln|7v+9| = x + C$$

Substitute $v = 2x + 3y$ back into the above equation to get an implicit solution of y:

$$14(2x+3y) - 39\ln|7(2x+3y)+9| = 49x + C_1 \quad (C_1 = 49C)$$

► TIP

Sometimes it is unnecessary or even impossible to find an explicit expression for the solution. An implicit solution is acceptable as long as it is reasonably simplified.

Linear First-Order Differential Equations

A first-order linear differential equation can be generally expressed as $\frac{dy}{dx} + p(x)\,y = q(x)$. This equation is not directly separable, but can be converted into a separable equation by multiplying both sides by an **integrating factor** $I(x)$. Then the equation becomes

$$I(x)\,y' + p(x)\,I(x)\,y = q(x)\,I(x)$$

To find $I(x)$, first notice that $\frac{d}{dx}(I(x)\,y) = I'(x)\,y + I(x)\,y'$, which resembles the left side of the previous equation. Let

$$I'(x)\,y + I(x)\,y' = I(x)\,y' + p(x)\,I(x)\,y$$

$$I'(x)\,y = p(x)\,I(x)\,y$$

$$\frac{d}{dx}I(x) = p(x)\,I(x)$$

$$\frac{d(I(x))}{I(x)} = p(x)\,dx$$

Integrating both sides,

$$\ln|I(x)| = \int p(x)\,dx$$

Since $I(x)$ is used as an integrating factor, there is no need to add a constant C here.

$$I(x) = e^{\int p(x)dx}$$

So the original equation with the integrating factor becomes

$$y'e^{\int p(x)dx} + p(x)\,ye^{\int p(x)dx} = q(x)\,e^{\int p(x)dx}$$

or

$$\frac{d}{dx}\left(ye^{\int p(x)dx}\right) = q\left(x\right)e^{\int p(x)dx}$$

which can be separated and solved analytically to obtain

$$y = e^{-\int p(x)dx}\left(\int q\left(x\right)e^{\int p(x)dx}dx + C\right)$$

► **NOTE**

The purpose of multiplying the integrating factor $I\left(x\right)$ is to make the left side of the equation a derivative with respect to x. Although it is generally quite difficult or even impossible to find an integration factor for a differential equation, you do not have to struggle every time with a linear first-order differential equation; you can directly apply the general solution formula to solve it.

Example 2.3

Solve the differential equation $\frac{dy}{dx} + x^2y = x^2$

Solution:

In the above equation, $p\left(x\right) = q\left(x\right) = x^2$, and $I\left(x\right) = e^{\int x^2dx} = e^{\frac{x^3}{3}}$. So the solution can be directly calculated as

$$y = e^{-\frac{x^3}{3}}\left(\int x^2e^{\frac{x^3}{3}}dx + C\right)$$

$$y = e^{-\frac{x^3}{3}}\left(\int e^u du + C\right) \quad (u = \frac{x^3}{3} \text{ and } du = x^2dx)$$

$$y = e^{-\frac{x^3}{3}}\left(e^{\frac{x^3}{3}} + C\right) = 1 + Ce^{-\frac{x^3}{3}}$$

Example 2.4

Solve the differential equation $\frac{dy}{dx} + 2y\cot x + \sin 2x = 0$.

Solution:

In the above equation, $p\left(x\right) = 2\cot x$, $q\left(x\right) = -\sin 2x$, and $I\left(x\right) = e^{\int 2\cot x dx}$. Letting $u = \sin x$ and $du = \cos x dx$, $I\left(x\right) = e^{2\int \frac{1}{u}du} = e^{2\ln|\sin x|} = \sin^2 x$. So the solution is

$$y = \frac{1}{\sin^2 x}\left(\int -\sin 2x\sin^2 x dx + C\right)$$

$$y = \frac{1}{\sin^2 x}\left(\int -2\sin x\cos x\sin^2 x dx + C\right)$$

$$y = \frac{1}{\sin^2 x}\left(-\int u du + C\right) \quad (u = \sin^2 x \text{ and } du = 2\sin x\cos x dx)$$

$$y = \frac{1}{\sin^2 x}\left(-\frac{\sin^4 x}{2} + C\right) = -\frac{\sin^2 x}{2} + \frac{C}{\sin^2 x}$$

Practice problem set 2

Solve the following differential equations.

1. $y^2 dx - x^2 dy = 0$
2. $\frac{dy}{dx} = \frac{2x-y}{x}$
3. $(x^3 + y^3)\, dx - 3xy^2 dy = 0$
4. $xdy - ydx - \sqrt{x^2 - y^2}dx = 0$
5. $\frac{dy}{dx} = \frac{2x+6y+3}{x+3y-9}$
6. $\frac{dy}{dx} + 2xy = 6x$
7. $(x-2)\frac{dy}{dx} = y + 4(x-2)^3$
8. $\frac{dy}{dx} + 2xy = 2x^3$; $y(0) = 1$
9. $\frac{dy}{dx} + y\cot x = 5e^{\cos x}$; when $x = \frac{\pi}{2}$, $y = -4$
10. $xy' = y(1 - x\tan x) + 2x^2 \cos x$

Chapter 3

Exponential Growth and Decay

Purpose: *To solve the differential equation for the exponential growth and decay model and to apply the solution. This is required by the AP Calculus AB/BC exams.*

So far we have learned how to solve separable differential equations and several other types that can be reduced to separable equations. In the following chapters we are going to apply our skills in solving some real world problems, namely, models of exponential growth and decay, simple inhibited growth and logistic growth.

Exponential growth and decay can be represented by one of the basic forms of separable differential equations:

$$\frac{dP}{dt} = rP \tag{3.1}$$

(Differential equation for exponential growth and decay)

This equation states that variable P varies at a rate directly proportional to the value of P. In the equation r is the rate constant. When $r > 0$, the equation represents exponential growth; when $r < 0$, it represents exponential decay.

Exponential growth and decay are most commonly used to model the change of a population, as the rate of population growth or decay is usually proportional to the population itself. For example, as a bacteria population increases, there are more individuals capable of reproduction so the rate of increase also increases. The decay of a radioactive substance also follows this law. As the amount of substance decreases, the rate of decay (the change of substance capable of producing radiation) also decreases. Exponential growth and decay can also be used to model an investment with interest compounded continuously, the processes of a solution being diluted by fresh water, a capacitor being discharged, and many other cases.

Equation 3.1 can be easily solved by separating the variables:

$$\frac{dP}{dt} = rP$$

$$\int \frac{dP}{P} = \int r dt$$

$$\ln|P| = rt + C$$

Since the population P is non-negative,

$$P = e^{rt+C} = C_1 e^{rt} \quad (C_1 = e^C)$$

Assuming that the initial value P_0 occurs at time $t = 0$, then $C_1 = P_0$. So we obtain this general solution:

$$P = P_0 e^{rt} \tag{3.2}$$

(Exponential growth and decay solution)

The following figure shows some examples of the exponential growth and decay function. In the graph it can be seen that if $r > 0$, $P \to \infty$ as $t \to \infty$; if $r < 0$, then $P \to 0$ as $t \to \infty$.

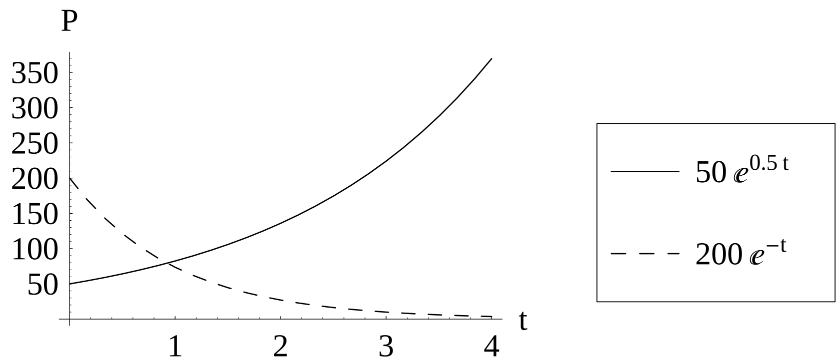


Figure 3.1: Examples of exponential growth and decay

Example 3.1

Bacteria in a culture increased from 400 to 1600 in three hours. Assuming that the rate of increase is directly proportional to the population,

a) Find an appropriate equation to model the population (assuming $P_0 = 400$ at time $t = 0$).

b) Find the number of bacteria at the end of six hours ($t = 6$) using the equation found above.

Solution:

a) Apply the initial condition $P_0 = 400$ in equation 3.2 to obtain $P = 400e^{rt}$. Since the population is 1600 in three hours, substitute $P = 1600$ and $t = 3$ in the solution equation to solve for e^r:

$$1600 = 400e^{3r}$$

$$e^{3r} = 4$$

$$e^r = 4^{\frac{1}{3}}$$

Now substitute the value for e^r back into the solution equation to get

$$P = 400\left(4^{\frac{t}{3}}\right)$$

b) Substituting $t = 6$ into the above equation, we have

$$P = 400\left(4^{\frac{6}{3}}\right) = 400\,(16) = 6400$$

► **TIP**

For the exponential growth/decay or other similar problems, it is often better to solve for e^r rather than r since the result can be obtained faster by using e^r directly.

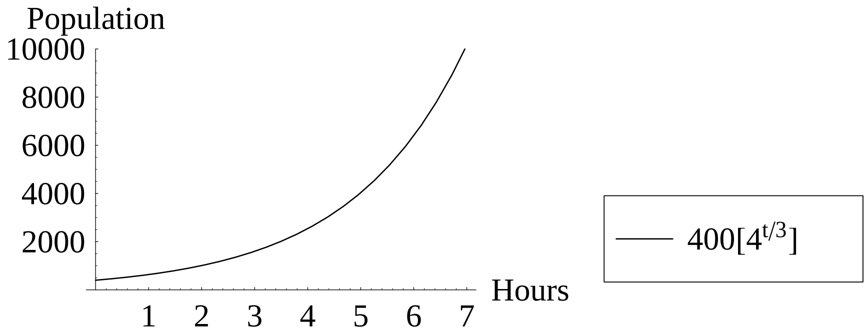


Figure 3.2: Graph for example 3.1

Example 3.2
Carbon-14 has a half life of approximately 5730 years (every 5730 years, the amount of radioactive substance will be halved). It is often used in carbon dating to find the age of artifacts and fossils since the amount of carbon-14 in the atmosphere is known. Assume that a certain fossil has 30% as much carbon-14 as its present-day equivalent should have. Approximate the age of the fossil.

Solution:
We first use the known condition to solve for e^r:

$$\frac{1}{2}P_0 = P_0 e^{5730r}$$

$$\frac{1}{2} = e^{5730r}$$

$$e^r = \left(\frac{1}{2}\right)^{\frac{1}{5730}}$$

Now use the percentage of radioactive material remaining to solve for t:

$$0.30 = \frac{P}{P_0} = e^{rt} = \left(\frac{1}{2}\right)^{\frac{t}{5730}}$$

$$\ln 0.30 = \ln\left[\left(\frac{1}{2}\right)^{\frac{t}{5730}}\right]$$

$$t = 5730\frac{\ln 0.30}{\ln 0.5} \approx 9953 \text{ years}$$

► TIP

As in the previous example, the general solution for half life problems is $P = P_0\left(\frac{1}{2}\right)^{\frac{t}{H}}$, where H is the half life.

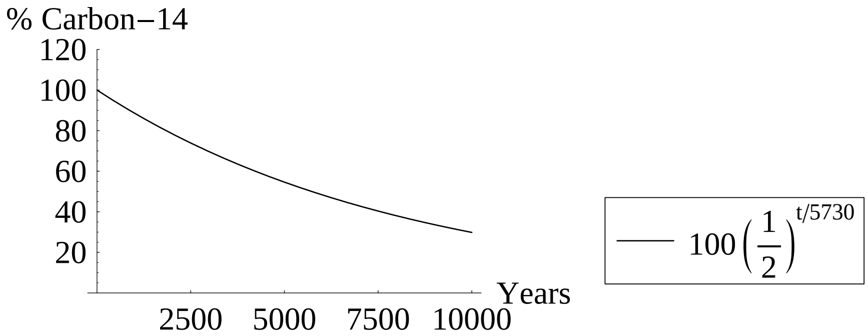


Figure 3.3: Graph for example 3.2

Example 3.3

Find the amount of money in a bank account with a 4% interest rate after 10 years if originally there was $5000 in it.

Solution:

Calculating money gained from interest is done directly with equation 3.2, where r is the interest rate and P_0 is the principal. In this case the equation is

$$P = 5000e^{0.04t}$$

Therefore the solution is

$$P = 5000e^{0.04(10)} \approx \$7459.12$$

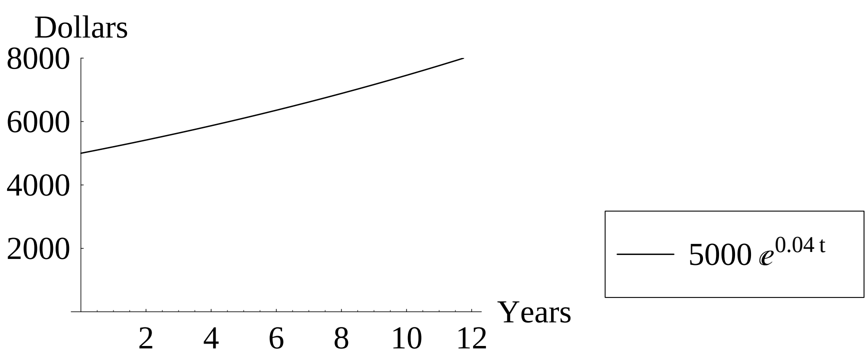


Figure 3.4: Graph for example 3.3

Practice problem set 3

Solve the following differential equations.

1. There are 120 grams of a radioactive substance whose half-life is 74 years. Determine how much of the substance will remain after 50 years.

2. A bacteria culture containing 2,400 cells 3 hours ago has now grown to 5,200 cells. Assuming the rate of growth is proportional to the number present, determine the time (from now) at which the population will reach 10,000.
3. A colony of bacteria with originally 100 cells has now grown to 400 in 2 hours. Assuming the rate of growth is proportional to the present number, find the number of bacteria in 5 hours from now.
4. There were 100 grams of a radioactive substance ten years ago, now there are only 32 grams. Find the substance's half-life.
5. A bank account contains $1,200 and pays an annual interest rate of 5.25% compounded continuously. Determine the time at which the money doubles.
6. An object is 3400 years old, find the percentage of its original Carbon-14 content it should have now (Carbon-14 has a half-life of 5730 years).
7. Intensity of light beam passing through an absorbing medium decreases at a rate proportional to the intensity at any given depth. Suppose at the surface of the water, the intensity of a light bulb is 20 candelas and 14 candelas under a yard of water. Find the light intensity under 20 feet of water.
8. The population of a country is growing at a rate proportional to its population. If the growth rate per year is 5% of the current population, in how many years will the population double?
9. Suppose the prices for real estate grow exponentially. If a house was worth $100,000 five years ago, and is now worth $250,000, find the year (from now) in which the price will exceed $500,000.
10. A tank initially holds 120 gallon of a brine solution containing 5 lb of salt. At $t = 0$, fresh water is poured into the tank at the rate of 6 gal/min, while the well-stirred mixture leaves the tank at the same rate. Find the time required for half of the salt to leave the tank.
11. When a capacitor is being discharged, the equation describing the charge on one plate of the capacitor is

    $$R\frac{dq}{dt} + \frac{q}{C} = 0$$

    where q is the charge (C), R is the resistance of the circuit (Ω), C is the capacitance of the capacitor (F), and t is time in seconds. If a 5 mF capacitor with an intial charge of 3 μC is discharged through a 100 Ω resistor, find the time when 90% of the charge has been drained.
12. A RL circuit has a resistance of 2 Ω, an inductance of 5 H, and an initial current of 8 A. Find (a) the current in the circuit at any time t and (b) its current after 10 seconds. The equation describing the current in the circuit is

    $$L\frac{dI}{dt} + RI = 0$$

    where L is the inductance (H), I is the current (A), R is the resistance (Ω), and t is time in seconds.

Chapter 4

Simple Inhibited Growth

Purpose: *To solve the differential equation for the simple inhibited growth model and to apply the solution. This is required by the AP Calculus AB/BC exams.*

In the previous chapter we saw that for the exponential growth model $\frac{dP}{dt} = rP$, when $r > 0$, $P \to \infty$ as $t \to \infty$. Usually the growth of a quantity in the real world is not unlimited, so in many cases, the exponential growth model is unrealistic for a long period of time. Let us assume that a natural maximum exists such that the growth of a quantity cannot occur beyond it. We can modify the above equation to reflect such a condition:

$$\frac{dP}{dt} = r\left(K - P\right) \tag{4.1}$$

(Differential equation for simple inhibited growth)

This is called **simple inhibited growth**. From equation 4.1 we can see that the rate of growth is limited by a constant K, the maximum population or quantity. Sometimes K is called the **carrying capacity** or **equilibrium value**, which is always assumed to be positive in this book. If P starts less than K, the growth of P is positive (assuming $r > 0$) until P is equal to K, at which point the growth of P diminishes to 0. If P is greater than K, then the growth of P is negative, which means P will decrease until it reaches K.

Simple inhibited growth can model the sales of a newly advertised product, in which case there exists a maximum limit of the product sales. It can also model the processes of an object cooling down to a certain temperature or being dropped from a certain height with air resistance. Other cases include the processes of a solution being diluted by another of different concentration, a capacitor being charged, and certain learning patterns.

We can use separation of variables to solve equation 4.1:

$$\frac{dP}{dt} = r\left(K - P\right)$$
$$\int \frac{dP}{K - P} = \int r dt$$
$$\ln|K - P| = -rt + C$$

If $P < K$, then

$$K - P = e^{-rt+C} = C_1 e^{-rt} \quad (C_1 = e^C)$$
$$P = K - C_1 e^{-rt}$$

Assume that the initial value P_0 occurs at time $t = 0$, then $C_1 = K - P_0$. So we obtain

$$P = K - \left(K - P_0\right) e^{-rt}$$

If $P > K$, then

$$P - K = e^{-rt+C} = C_1 e^{-rt}$$

$$P = K + C_1 e^{-rt}$$

In this case $C_1 = P_0 - K$, so we still have the same equation:

$$P = K - (K - P_0)\, e^{-rt} \tag{4.2}$$

(Simple inhibited growth solution)

The figure below shows some examples of simple inhibited growth. From the graph it can be seen that when $r > 0$, $P \to K$ as $t \to \infty$, regardless of the choice of P_0.

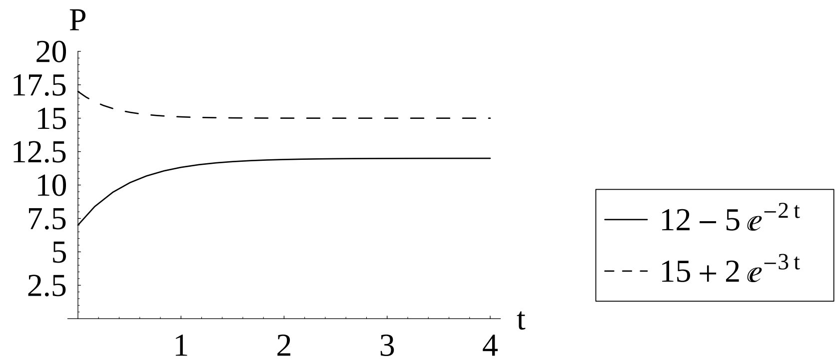


Figure 4.1: Simple inhibited growth ($r > 0$)

However, when $r < 0$, $e^{-rt} \to \infty$ as $t \to \infty$, therefore $P \to \pm\infty$, depending on the sign of $(K - P_0)$. The following figure shows two such examples.

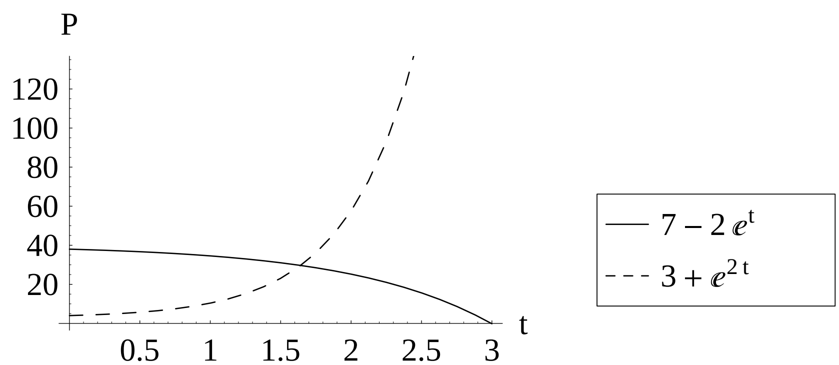


Figure 4.2: Simple inhibited growth ($r < 0$)

Example 4.1

After an advertisement for a product has being aired for the first time on television, it is predicted that at most 35% of the population will purchase it. From sales records, 15% of the population has already purchased it after 14 days.

a) Find an appropriate equation to model the sales.

b) Approximate the number of days it will take for 34% of the population to have purchased it.

Solution:

a) Since the advertisement was aired for the first time, $P_0 = 0$ and equation 4.2 becomes:

$$P = K\left(1 - e^{-rt}\right)$$

Now substitute the values of K (35), P (15) and t (14) into the above equation:

$$15 = 35\left(1 - e^{-14r}\right)$$

$$e^{-14r} = 1 - \frac{15}{35} = \frac{4}{7}$$

$$e^{-r} = \left(\frac{4}{7}\right)^{\frac{1}{14}}$$

With this information, the equation is complete as

$$P = 35\left(1 - \left(\frac{4}{7}\right)^{\frac{t}{14}}\right)$$

b) Substituting $P = 34$ to solve for t,

$$34 = 35\left(1 - \left(\frac{4}{7}\right)^{\frac{t}{14}}\right)$$

$$\frac{1}{35} = \left(\frac{4}{7}\right)^{\frac{t}{14}}$$

$$t = 14\frac{\ln\frac{1}{35}}{\ln\frac{4}{7}} = -14\left(\frac{\ln 35}{\ln 4 - \ln 7}\right) \approx 89 \text{ days}$$

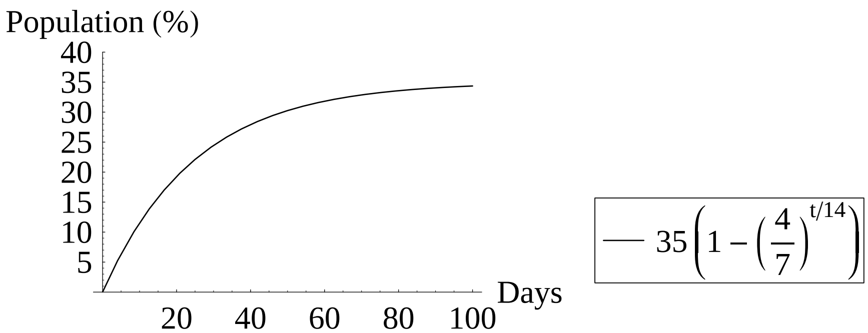


Figure 4.3: Graph for example 4.1

Example 4.2
According to Newton's law of cooling, the rate at which an object cools (or warms) is directly proportional to the temperature difference between the environment and the object itself. If a pot of boiling water (100°C) is left at room temperature (22°C) and after five minutes the water is only 70°C, find its temperature after another 5 minutes.

Solution:
Substituting the values of P_0 (100), K (22), P (70), and t (5) into equation 4.2,

$$70 = 22 - (22 - 100)\, e^{-5r}$$

$$\frac{70 - 22}{78} = e^{-5r}$$

$$e^{-r} = \left(\frac{8}{13}\right)^{\frac{1}{5}}$$

Therefore after another 5 minutes,

$$T = 22 + 78\left(\frac{8}{13}\right)^{\frac{10}{5}} \approx 51°\text{C}$$

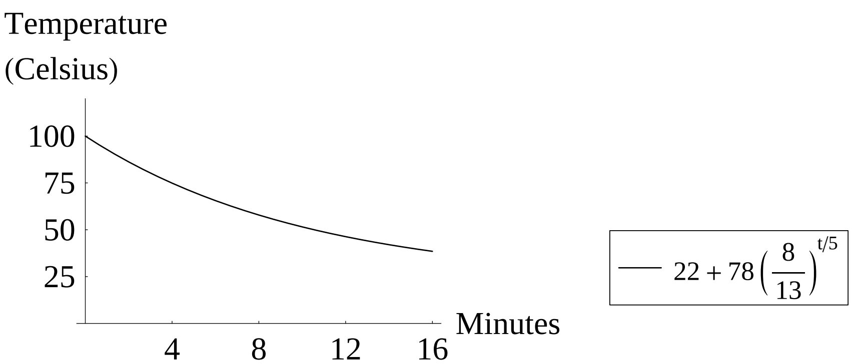


Figure 4.4: Graph for example 4.2

Example 4.3
A tank initially holds 100 gallon of brine solution containing 1 lb of salt. At $t = 0$ another brine solution containing 1 lb of salt per gallon is poured into the tank at the rate of 3 gal/min, while the well-stirred mixture leaves the tank at the same rate. Find the amount of salt in the tank at any time.

Solution:
Assume there is Q lb of salt in the tank at time t. The concentration of salt in the solution at time t is $\frac{Q}{100}$ lb/gal. The rate of salt being added to the tank is (1 lb/gal)(3 gal/min) = 3 lb/min. The

rate of salt leaving the tank is ($\frac{Q}{100}$ lb/gal)(3 gal/min) = 0.03Q lb/min. So the overall change rate of salt concentration is:

$$\frac{dQ}{dt} = 3 - 0.03Q = 0.03\,(100 - Q)$$

$$\int \frac{dQ}{100 - Q} = \int 0.03dt$$

$$-\ln|100 - Q| = 0.03t + C$$

$$100 - Q = e^{-0.03t - C}$$

$$Q = 100 - C_1 e^{-0.03t} \quad (C_1 = e^{-C})$$

Substituting the initial condition $Q\,(0) = 1$ into the above equation, we can solve for C_1:

$$C_1 = 100 - 1 = 99$$

So the amount of salt in the tank at time t (in minutes) is:

$$Q = 100 - 99e^{-0.03t} \text{ lbs.}$$

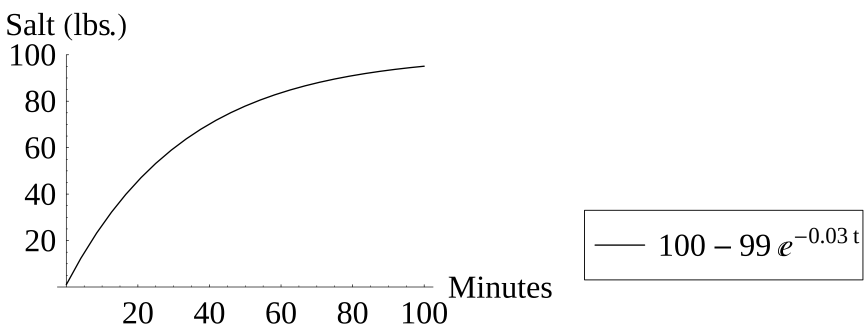


Figure 4.5: Graph for example 4.3

Practice problem set 4

Solve the following problems using the simple inhibited growth model.

1. Suppose survey results determined that no matter how long a product is advertised, no more than 30% of the population will buy it. After 12 days of advertising, 4% of the population has bought it. Approximate the time at which 20% of the population will have bought it.

2. Suppose a student can memorize a maximum of 50 words in a single attempt. After 15 minutes of memorization, the student can recall 20 words. Find the number of words this student can memorize in 30 minutes.

3. A glass of iced water (3°C) is left in the shade outside (27°C). After 5 minutes, the water's temperature is 7°C. Find the water's temperature in another ten minutes.

4. The resisting force against an object falling in air is proportional to its velocity, so eventually any object falling in air will reach a terminal velocity. If somehow a cinder block is dropped from rest in the sky, and after five seconds its velocity is 45 m/s, and after five more seconds its velocity is 80 m/s, find its terminal velocity.

5. Theoretically, if a car tire punctures, it will never equalize it's pressure with the atmosphere because the tire pressure decreases according to simple inhibited growth (decay in this case). Assuming the original tire pressure was 35 psi relative to the atmosphere, find how long it would take for the tire to deflate 99.9% if after 5 minutes, the pressure dropped to 12 psi.

6. The rate at which salt dissolves in water is directly proportional to the amount that remains undissolved. If 5 pounds of salt are placed in a container of water and 1 pound dissolves in 5 minutes, find how long it will take to dissolve another pound.

7. Suppose a corpse was discovered in a hotel room at noon and its temperature was 80°F. The room was kept at a constant 65°F, and now, after three hours, the temperature of the corpse was 72°F. Find the time of death assuming the body was originally at a temperature of 98.6°F.

8. Suppose a certain country's population has constant relative birth and death rates of 97 births per thousand people per year and 47 deaths per thousand people per year respectively. Assume also that approximately 30000 people emigrate from the country every year. What is the equation that models the population $P(t)$ of the country, where t is in years?

9. An 8 kg weight falls from rest towards the earth. Assuming that the weight is acted upon by air resistance, numerically equal to 2 times its speed but with units of newtons. Find the velocity of the weight fallen after t seconds. Hint: use Newton's second law $ma = F_{wt} - F_{air} = mg - 2v$ or $m\frac{dv}{dt} = (mg - 2v)$ ($g = 9.8\, {}^{m}/{}_{s^2}$).

10. A tank initially holds 80 gallon of a brine solution containing 2 lb of salt. At $t = 0$, another brine solution containing 1 lb of salt per gallon is poured into the tank at the rate of 4 gal/min, while the well-stirred mixture leaves the tank at the same rate. Find the amount of salt in the tank at any time.

11. When a capacitor is being charged, the equation governing the amount of charge on a plate is

$$R\frac{dq}{dt} + \frac{q}{C} = \mathcal{E}$$

where q is the charge (C), R is the resistance of the circuit (Ω), and C is the capacitance of the capacitor (F), $\mathcal{E}$ is the applied voltage (V), and t is time in seconds. Assuming 1.5 V is applied across an initially empty 1 mF capacitor in a circuit with resistance 10 kΩ. Find the amount of charge on one plate of the capacitor after 1 second.

12. In a circuit with a 50 kΩ resistor, 9 V is applied across a 500 μF capacitor with an initial charge of 1 mC. Find the time at which the charge of the capacitor reaches 3 mC.

Chapter 5

Logistic Growth

Purpose: *To solve the differential equation for the logistic growth model and to apply the solution. This is only required by the AP Calculus BC exam.*

In the previous two chapters, we have discussed cases in which the rate of change of quantity P is either directly proportional to itself (P), or to its remaining room for growth $(K - P)$. **Logistic growth** deals with growth rates that are directly proportional to both of these quantities:

$$\frac{dP}{dt} = r' P (K - P) \tag{5.1}$$

Here r' is used because the logistic equation is more commonly written in this form:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) \tag{5.2}$$

(Differential equation for logistic growth)

where $r = r'K$. In the above equation, K is the same carrying capacity or equilibrium value as we discussed before. The constant r is called the **intrinsic growth rate**, that is, the growth rate in the absence of any limiting factors. The logistic equation is mostly used to provide a more realistic model for population growth (refer to Appendix A for a detailed derivation). The logistic equation is also frequently used to describe the spreading of diseases or rumors, autocatalytic chemical reactions, and other processes.

The logistic equation shows that if P is small relative to the carrying capacity K, the rate of its growth will be close to the constant rate r of the exponential growth model. As P nears K, the rate will shrink toward 0, resulting in an S-shaped curve (refer to Figures 5.1 and 5.2). According to this model, when P reaches K, the growth rate is 0, and the population will be stable. If P were to somehow exceed K, the rate would become negative and the population would decrease toward K.

In order to solve equation 5.2, we separate the variables first and integrate both sides:

$$\int \frac{dP}{P\left(1 - \frac{P}{K}\right)} = \int \frac{K dP}{P (K - P)} = \int r dt$$

Separating the integrand by partial fractions we have

$$\frac{K}{P (K - P)} = \frac{1}{P} + \frac{1}{K - P}$$

Therefore,

$$\int \frac{dP}{P} + \int \frac{dP}{K - P} = \int r dt$$

$$\ln |P| - \ln |K - P| = rt + C$$

$$\ln\left|\frac{K-P}{P}\right| = -rt - C$$

If $\frac{K-P}{P} > 0$, we have:

$$\frac{K-P}{P} = C_1 e^{-rt} \quad (C_1 = e^{-C})$$

Assuming that $P = P_0$ when $t = 0$, then

$$C_1 = \frac{K-P_0}{P_0}$$

Therefore

$$\frac{K-P}{P} = \frac{K-P_0}{P_0} e^{-rt}$$

$$P = \frac{K}{1 + \frac{K-P_0}{P_0} e^{-rt}} = \frac{KP_0}{P_0 + (K-P_0)\, e^{-rt}}$$

If $\frac{K-P}{P} < 0$, we have:

$$\frac{P-K}{P} = C_1 e^{-rt} \quad (C_1 = e^{-C})$$

Assume that $P = P_0$ when $t = 0$, then

$$C_1 = \frac{P_0 - K}{P_0}$$

$$\frac{P-K}{P} = \frac{P_0-K}{P_0} e^{-rt}$$

This will lead to the same solution as in the previous case. So the final solution is:

$$P = \frac{KP_0}{P_0 + (K-P_0)\, e^{-rt}} \tag{5.3}$$

(Logistic growth solution)

Example 5.1

A population of bacteria in a culture is 50 million, and is growing at a rate of 2 million per hour. Assume the carrying capacity is 1 billion. Use one million as a base unit.

a) Write the logistic differential equation using the data.

b) Use the model to predict the population in 2 hours, 5 hours, and a day from now.

c) Use the model to predict when the population will reach half the carrying capacity.

Solution:

a) Since the initial population is small compared to the carrying capacity, take the initial relative growth rate ($\frac{2}{50}$) to be an estimate of r. Substitute all the known numbers into equation 5.2:

$$\frac{dP}{dt} = \frac{2}{50} P\left(1 - \frac{P}{1000}\right) = 0.04P\left(1 - \frac{P}{1000}\right)$$

b) Using equation 5.3,

$$P = \frac{KP_0}{P_0 + (K - P_0)\, e^{-rt}} = \frac{(1000)\,(50)}{50 + (1000 - 50)\, e^{-0.04t}} = \frac{1000}{1 + 19e^{-0.04t}}$$

$$P\,(2) \approx 53.9 \text{ million}$$

$$P\,(5) \approx 60.4 \text{ million}$$

$$P\,(24) \approx 121 \text{ million}$$

c) Set $P = 500$ to solve for t:

$$500 = \frac{1000}{1 + 19e^{-0.04t}}$$

$$19e^{-0.04t} = 1$$

$$t \approx 73.6 \text{ hours}$$

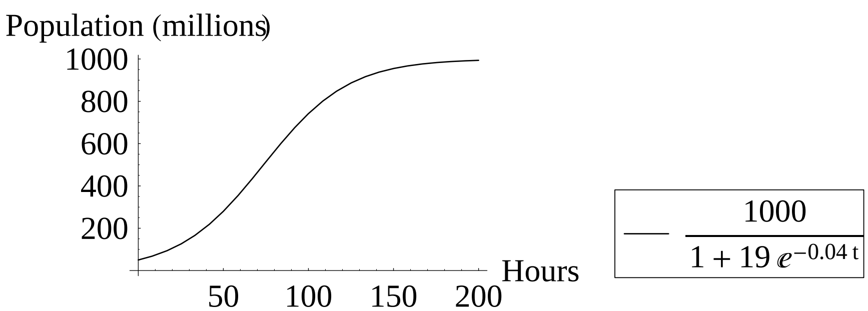


Figure 5.1: Graph for example 5.1

Example 5.2

A rumor is spreading in a city of 6000 people. Initially, three people know it; three days later 300 people have heard about it. Suppose the rumor spreads at a rate proportional to both the number of people knowing it and the number of people not knowing it. Find

a) the number of days for the rumor to spread to 50% of the people,

b) the approximate number of people knowing it after ten days.

Solution:

a) Use equation 5.3 to solve for e^{-r} with the known information $P_0 = 3$, $K = 6000$ and $P\,(3) = 300$.

$$300 = \frac{(6000)\,(3)}{3 + (6000 - 3)\, e^{-3r}}$$

$$3 + (6000 - 3)\, e^{-3r} = 60$$

$$1999e^{-3r} = 19$$

$$e^{-r} = \left(\frac{19}{1999}\right)^{\frac{1}{3}}$$

Now use the value for e^{-r} and $P = 3000$ to solve for t:

$$3000 = \frac{18000}{3 + 5997e^{-rt}}$$

$$3 + 5997e^{-rt} = 6$$

$$1999e^{-rt} = 1$$

$$\left(\frac{19}{1999}\right)^{\frac{t}{3}} = \frac{1}{1999}$$

$$t = \frac{3\ln\frac{1}{1999}}{\ln\frac{19}{1999}} \approx 4.9 \text{ days}$$

b) Letting $t = 10$, use equation 5.3 to solve for P:

$$P(10) = \frac{18000}{3 + 5997\left(\frac{19}{1999}\right)^{\frac{10}{3}}} \approx 5998 \text{ people}$$

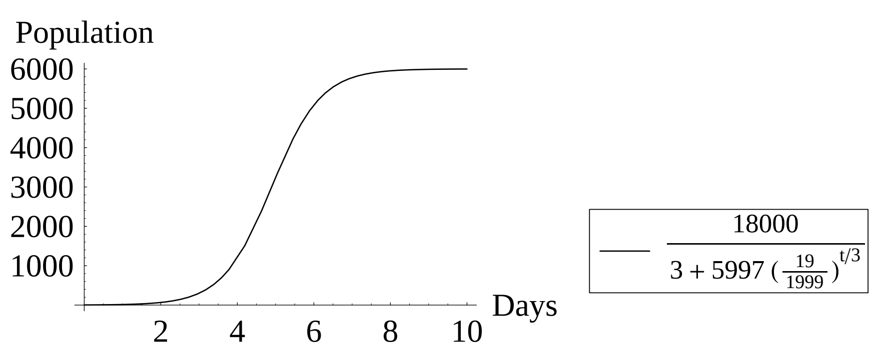


Figure 5.2: Graph for example 5.2

Practice problem set 5

Solve the following problems.

1. A certain population has 10,000 people. A disease is spreading through the population at a rate proportional to both the infected population and to the unaffected population. If it is known that 1,000 were infected two months ago, and 4,000 were infected last month, how many should be infected this month? How many months from now will 90% of the population be infected?

2. The number of people that hear a rumor follows logistic growth. In a school of 1500 students, 5 students start a rumor. After 2 hours, 120 students have heard about the rumor. Find the number of students to hear the rumor after one more hour.

3. Diseases sometimes spread according to logistic growth. If there were 100 people infected one week ago and 280 infected now out of an isolated population of 30,000, find the number of people that will be infected a week from today.

4. A salmon population of 1.5 million living off the coast of Alaska grows at a rate of $0.04P(t)$ per year, where $P(t)$ is the salmon population at time t. Suppose a group of predators moves into the waters of the salmon and starts to kill the salmon at a rate of $0.0002[P(t)]^2$ per year. Calculate how large the salmon population is after 5 years.

5. The rate of formation of a certain chemical X in the second order chemical reaction $A + B \to X$ is known to be governed by the equation

$$\frac{dx}{dt} = r(a-x)(b-x) \qquad (r > 0, a > b > 0)$$

where x is the amount (concentration) of chemical X present at time t, and a, b are the initial concentrations of A and B, respectively. If $x = \frac{1}{2}(a+b)$ when $t = 0$, find x as a function of t and determine $\lim_{t\to\infty} x(t)$. Hint: use Example 1.4.

6. A rumor is spreading in a population of 800. Assume that each person meets four people each day (who may or may not know the rumor). Initially one person knows the rumor.

   a) When will 200 people know it?

   b) When will 799 people know it?

   Hint: the intrinsic growth rate is $r = 4$.

7. It is known that the enzyme pepsin digests protein in the stomach and is formed from the cleavage of a short segment of amino acids from its predecessor, pepsinogen. Under conditions of extremely low pH (the pH of the stomach is around 2), pepsin can also convert pepsinogen into pepsin, thus setting up an autocatalytic reaction. Assuming that there is 1.5 M pepsin in the stomach prior to a meal, and 5 M pepsinogen is secreted, find when 99% of the newly available pepsinogen will be converted, if after 10 seconds, there exists 2 M pepsin. Hint: use the logistic equation with $P_0 = 1.5$.

8. There are two islands, A and B. Initially, there are 1000 people on island A and no one on island B. If the rate of emigration is proportional to the difference between the population of A and the population of B, and after 5 years there are 250 people on island B and 750 people on island A, how many years does it take for island B to have 400 people? Hint: is this logistic growth?

Chapter 6

Implicit Equation Forms*

Purpose: *To manipulate the previous three growth modeling equations into implicit and symmetric forms. The material in this chapter is not covered explicitly on the AP Calculus exams.*

In this chapter we are going to show that all three of the previous growth modeling equations, plus a fourth model which is often used to describe certain chemical reactions, can be written in highly symmetric and implicit forms[1] for easy memorization and comparison. These implicit forms can help you better understand the nature of these differential equations and provide you certain advantages in calculating the solutions.

These implicit forms are similar to the exponential growth and decay equation and its solution:

$$\frac{dP}{dt} = rP \tag{3.1}$$

$$P = P_0 e^{rt} \tag{3.2}$$

Let us first review the simple inhibited growth equations:

$$\frac{dP}{dt} = r\left(K - P\right) \tag{4.1}$$

$$P = K - \left(K - P_0\right) e^{-rt} \tag{4.2}$$

If we make the substitution $Z = K - P$, then $\frac{dZ}{dt} = -\frac{dP}{dt}$, so equation 4.1 becomes $\frac{dZ}{dt} = -rZ$. This new equation of Z resembles the form of equation 3.1 except that the rate constant is $-r$. We can directly solve for Z by using equation 3.2 and changing the sign of r:

$$Z = Z_0 e^{-rt}$$

where $Z = K - P$ and $Z_0 = K - P_0$. Substituting these into the above equations of Z, we have:

$$\frac{d\left(K - P\right)}{dt} = -r\left(K - P\right) \tag{6.1}$$

$$K - P = \left(K - P_0\right) e^{-rt} \tag{6.2}$$

(Implicit forms of simple inhibited growth)

Equations 6.1 and 6.2 can be easily converted back to equations 4.1 and 4.2, respectively.

Next, we work on the logistic growth equations:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) \tag{5.2}$$

$$P = \frac{KP_0}{P_0 + \left(K - P_0\right) e^{-rt}} \tag{5.3}$$

[1] Professor Kuo Chen, Principal of Olympia Institute in San Francisco, summarized the first three growth equations into implicit forms. A portion of this chapter is adapted from his lecture notes.

We make the substitution $Z = \frac{K-P}{P}$, then $Z = \frac{K}{P} - 1$, or $P = \frac{K}{Z+1}$, and also $\frac{dZ}{dt} = -\frac{K}{P^2}\frac{dP}{dt}$ or $\frac{dP}{dt} = -\frac{P^2}{K}\frac{dZ}{dt}$. Substitute these into equation 5.2 and rearrange:

$$-\frac{P^2 dZ}{K dt} = rP\left(1 - \frac{P}{K}\right)$$

$$\frac{dZ}{dt} = -\frac{rK}{P}\left(1 - \frac{P}{K}\right)$$

Substituting $P = \frac{K}{Z+1}$ into the above equation, we obtain

$$\frac{dZ}{dt} = -r\left(Z+1\right)\left(1 - \frac{1}{Z+1}\right)$$

$$\frac{dZ}{dt} = -rZ$$

Again, we can directly obtain the solution of Z:

$$Z = Z_0 e^{-rt}$$

Substituting $Z = \frac{K-P}{P}$ and $Z_0 = \frac{K-P_0}{P_0}$ into the above two equations of Z, we have:

$$\frac{d}{dt}\left(\frac{K-P}{P}\right) = -r\left(\frac{K-P}{P}\right) \tag{6.3}$$

$$\frac{K-P}{P} = \frac{K-P_0}{P_0}e^{-rt} \tag{6.4}$$

(Implicit forms of logistic growth)

With a little bit of manipulation, the above implicit forms can be converted to equations 5.2 and 5.3, respectively.

We can also rewrite the exponential growth and decay equations in implicit forms so that they match the other two types of growth equations in format. Let $Z = \frac{1}{P}$, then $\frac{dZ}{dt} = -\frac{1}{P^2}\frac{dP}{dt}$ or $\frac{dP}{dt} = -P^2\frac{dZ}{dt}$. Substitute these into equation 3.1:

$$-P^2\frac{dZ}{dt} = rP$$

$$\frac{dZ}{dt} = -r\frac{1}{P} = -rZ$$

The solution of Z is:

$$Z = Z_0 e^{-rt}$$

Since $Z = \frac{1}{P}$, $Z_0 = \frac{1}{P_0}$, we have obtained the following:

$$\frac{d}{dt}\left(\frac{1}{P}\right) = -r\left(\frac{1}{P}\right) \tag{6.5}$$

$$\frac{1}{P} = \frac{1}{P_0}e^{-rt} \tag{6.6}$$

(Implicit forms of exponential growth and decay)

Now we will study a fourth type of growth modeling equation which is frequently used for describing the processes of second order chemical reactions. Because of this we are going to call it **second order growth** in this book. The equation is listed in practice problem 5 of the previous chapter and shown here:

$$\frac{dP}{dt} = r\left(a-P\right)\left(b-P\right) \qquad (a \neq b) \tag{6.7}$$

This equation states that the rate of change of quantity P is directly proportional to its remaining room for growth within the limits a and b. The solution to this equation can be derived easily from Example 1.4. With the addition of a rate constant r, the explicit solution is

$$P = a + \frac{\left(b-a\right)\left(P_0-a\right)}{\left(P_0-a\right)+\left(b-P_0\right)e^{r(b-a)t}} \tag{6.8}$$

Now we will make a substitution similar to the previous ones to derive the implicit forms of equations 6.7 and 6.8. Let $Z = \frac{b-P}{a-P}$, then

$$aZ - PZ = b - P$$

$$P = \frac{aZ-b}{Z-1}$$

Therefore, the differentials are:

$$\frac{dZ}{dt} = \frac{-\left(a-P\right)+\left(b-P\right)}{\left(a-P\right)^2}\frac{dP}{dt} = \frac{b-a}{\left(a-P\right)^2}\frac{dP}{dt}$$

$$\frac{dP}{dt} = \frac{\left(a-P\right)^2}{b-a}\frac{dZ}{dt}$$

Substituting all of these into equation 6.7, we obtain

$$\frac{\left(a-P\right)^2}{b-a}\frac{dZ}{dt} = r\left(a-\frac{aZ-b}{Z-1}\right)\left(b-\frac{aZ-b}{Z-1}\right)$$

$$\frac{\left(a-P\right)^2}{b-a}\frac{dZ}{dt} = r\left(\frac{aZ-a-aZ+b}{Z-1}\right)\left(\frac{bZ-b-aZ+b}{Z-1}\right) = r\left(\frac{b-a}{Z-1}\right)\left(\frac{bZ-aZ}{Z-1}\right) = r\frac{Z\left(b-a\right)^2}{\left(Z-1\right)^2}$$

Since $a-P = a-\frac{aZ-b}{Z-1} = \frac{aZ-a-aZ+b}{Z-1} = \frac{b-a}{Z-1}$, we can easily substitute away the $\left(a-P\right)^2$ on the left side of the above equation:

$$\frac{\left(\frac{b-a}{Z-1}\right)^2}{b-a}\frac{dZ}{dt} = r\frac{Z\left(b-a\right)^2}{\left(Z-1\right)^2}$$

$$\frac{1}{b-a}\frac{dZ}{dt} = rZ$$

$$\frac{dZ}{dt} = r(b-a)Z$$

So the solution for Z is:

$$Z = Z_0 e^{r(b-a)t}$$

Substituting $Z = \frac{b-P}{a-P}$ and $Z_0 = \frac{b-P_0}{a-P_0}$ into the above equations we have:

$$\frac{d}{dt}\left(\frac{b-P}{a-P}\right) = r(b-a)\left(\frac{b-P}{a-P}\right) \qquad (a \neq b) \tag{6.9}$$

$$\frac{b-P}{a-P} = \left(\frac{b-P_0}{a-P_0}\right)e^{r(b-a)t} \tag{6.10}$$

(Implicit forms of second order growth)

The implicit solution found above can be transformed into the explicit solution with only a few manipulations.

► **NOTE**

The implicit model equations derived above can better reveal the nature of these models. For example, in the case of $r > 0$, equation 6.1 shows that the simple inhibited growth is an exponential decay of the $(K-P)$ quantity; equation 6.3 shows that the logistic growth is an exponential decay of the $\left(\frac{K-P}{P}\right)$ quantity; and equation 6.9 shows that the second order growth is an exponential growth of the $\left(\frac{b-P}{a-P}\right)$ quantity if $b > a$, and exponential decay if $b < a$. A summary of all the implicit forms is provided in the inside front cover of the book.

Example 6.1

Solve Example 5.2a using the implicit solution.

Solution:

Using the conditions $P_0 = 3$ and $P(3) = 300$ in equation 6.4 we have:

$$\frac{6000-300}{300} = \frac{6000-3}{3}e^{-r(3)}$$

$$e^{-r} = \left(\frac{19}{1999}\right)^{\frac{1}{3}}$$

Solving for t when $P(t) = 3000$:

$$\frac{6000-3000}{3000} = \frac{6000-3}{3}e^{-rt}$$

$$1 = 1999\left(\frac{19}{1999}\right)^{\frac{t}{3}}$$

$$t = \frac{3 \ln \frac{1}{1999}}{\ln \frac{19}{1999}} \approx 4.9 \text{ days}$$

Example 6.2

Shown below is an example of a bimolecular nucleophilic substitution of a primary halide:

$$\mathrm{NaOH} + \mathrm{CH_3Br} \to \mathrm{CH_3OH} + \mathrm{NaBr}$$

This is an example of second order chemical reaction where the concentrations of both reactants affect the reaction rate. Letting x represent the concentration of $\mathrm{CH_3OH}$, find when the reaction has reached the half way point, i.e. x reaches 50% of its final amount. Assume at time $t = 0$ there are 1.0 M NaOH and 1.2 M $\mathrm{CH_3Br}$ and the reaction rate constant r is known to be 0.02 M/s.

Solution:

A complete reaction would exhaust the lesser quantity (in this case 1.0 M NaOH) and produce 1 M $\mathrm{CH_3OH}$, so the half way point of the reaction is when 0.5 M $\mathrm{CH_3OH}$ has been produced. Using equation 6.10, the implicit solution of second order growth:

$$\frac{1.2 - 0.5}{1.0 - 0.5} = \frac{1.2 - 0}{1.0 - 0} e^{0.02(1.2 - 1.0)t}$$

$$\frac{7}{6} = e^{0.004t}$$

$$t = 250 \ln \frac{7}{6} \approx 38.5 \text{ s}$$

Example 6.3

The following table shows the fish population in a lake in three consecutive decades. Assume that the population grows logistically. Estimate the lake's maximum capacity for the fish population.

Year	1960	1970	1980
Population (thousands of fish)	61.3	72.8	84.0

Solution:

In this problem we know $P_0 = 61.3$, $P_1 = 72.8$, $P_2 = 84.0$, $t_0 = 0$, $t_1 = 1$, $t_2 = 2$ and want to find K. We will first derive a general solution for K using the implicit forms, and then plug in the known numbers to solve for K. According to equation 6.4 we have:

$$\frac{K - P_1}{P_1} = \left(\frac{K - P_0}{P_0}\right) e^{-rt_1} \text{ and } \frac{K - P_2}{P_2} = \left(\frac{K - P_0}{P_0}\right) e^{-rt_2}$$

Divide the first equation by the second:

$$\frac{(K - P_1) P_2}{(K - P_2) P_1} = e^{r(t_2 - t_1)}$$

Rearranging the first equation:

$$\frac{(K-P_0)\,P_1}{(K-P_1)\,P_0}=e^{rt_1}$$

Since $t_2-t_1=t_1=1$, then $e^{rt_1}=e^{r(t_2-t_1)}$, so we have:

$$\frac{(K-P_0)\,P_1}{(K-P_1)\,P_0}=\frac{(K-P_1)\,P_2}{(K-P_2)\,P_1}$$

Solving the above quadratic equation of K:

$$(K-P_0)\,(K-P_2)\,P_1^2=(K-P_1)^2\,P_0P_2$$

$$\left(P_1^2-P_0P_2\right)K^2+\left(2P_0P_1P_2-P_0P_1^2-P_2P_1^2\right)K=0$$

$$K_1=0,\;K_2=\frac{P_1\,(P_1P_0+P_1P_2-2P_0P_2)}{P_1^2-P_0P_2}$$

Obviously, $K_1=0$ is not a meaningful solution, so the solution is:

$$K=\frac{72.8\,(72.8\,(61.3)+72.8\,(84.0)-2\,(61.3)\,(84.0))}{72.8^2-(61.3)\,(84.0)}\approx 135.0 \text{ thousand}$$

Example 6.4

The census taken in 1990 and 1994 of a city's population showed that it had 2.48 million and 2.67 million residents, respectively. Assume that the maximum capacity of the city is 3.20 million and the population increases logistically. Estimate the city's population in 1978.

Solution:

In this problem we know $P_1=2.48$, $P_2=2.67$, $t_1=12$, $t_2=16$, $K=3.20$, and we want to find P_0. Again we will derive a general solution of P_0, and then plug in the known numbers to solve for it. From the previous example we have:

$$\frac{(K-P_0)\,P_1}{(K-P_1)\,P_0}=e^{rt_1}\text{ and }\frac{(K-P_1)\,P_2}{(K-P_2)\,P_1}=e^{r(t_2-t_1)}$$

Solve for e^r in both equations and equate them:

$$\left(\frac{(K-P_0)\,P_1}{(K-P_1)\,P_0}\right)^{\frac{1}{t_1}}=\left(\frac{(K-P_1)\,P_2}{(K-P_2)\,P_1}\right)^{\frac{1}{t_2-t_1}}$$

$$\frac{(K-P_0)\,P_1}{(K-P_1)\,P_0}=\left(\frac{(K-P_1)\,P_2}{(K-P_2)\,P_1}\right)^{\frac{t_1}{t_2-t_1}}$$

$$\frac{K-P_0}{P_0}=\frac{K-P_1}{P_1}\left(\frac{(K-P_1)\,P_2}{(K-P_2)\,P_1}\right)^{\frac{t_1}{t_2-t_1}}$$

$$\frac{K}{P_0} = 1 + \frac{K - P_1}{P_1}\left(\frac{(K - P_1)\, P_2}{(K - P_2)\, P_1}\right)^{\frac{t_1}{t_2 - t_1}}$$

$$P_0 = \frac{K}{1 + \frac{K-P_1}{P_1}\left(\frac{(K-P_1)P_2}{(K-P_2)P_1}\right)^{\frac{t_1}{t_2-t_1}}}$$

$$P_0 = \frac{KP_1}{P_1 + (K - P_1)\left(\frac{(K-P_1)P_2}{(K-P_2)P_1}\right)^{\frac{t_1}{t_2-t_1}}}$$

So the final solution is:

$$P_0 = \frac{(3.20)\,(2.48)}{2.48 + (3.20 - 2.48)\left(\frac{(3.20-2.48)2.67}{(3.20-2.67)2.48}\right)^{\frac{12}{4}}} \approx 1.68 \text{ million}$$

► **NOTE**

Examples 6.3 and 6.4 show that the implicit equation forms can facilitate the derivation of certain parameter expressions, such as K and P_0 in the logistic equation. The same method can be applied to derive the expressions of a, b and P_0 in the second order growth equation, as shown by example 6.5. These expressions are summarized in the back cover of this book.

Example 6.5

For the second order chemical reaction described by:

$$\frac{dP}{dt} = r\,(a - P)\,(b - P) \qquad (a \neq b)$$

express parameter b in terms of a, P_0, P_1, P_2, t_1 and t_2, under the condition $t_2 - t_1 = t_1$.

Solution:

According to equation 6.10 we have:

$$\frac{b - P_1}{a - P_1} = \left(\frac{b - P_0}{a - P_0}\right) e^{r(b-a)t_1} \text{ and } \frac{b - P_2}{a - P_2} = \left(\frac{b - P_0}{a - P_0}\right) e^{r(b-a)t_2}$$

Divide the second equation by the first:

$$\frac{(b - P_2)\,(a - P_1)}{(b - P_1)\,(a - P_2)} = e^{r(b-a)(t_2-t_1)}$$

Rearranging the first equation:

$$\frac{(b - P_1)\,(a - P_0)}{(a - P_1)\,(b - P_0)} = e^{r(b-a)t_1}$$

Since $t_2 - t_1 = t_1$, then $e^{r(b-a)(t_2-t_1)} = e^{r(b-a)t_1}$, so we have:

$$\frac{(b-P_2)(a-P_1)}{(b-P_1)(a-P_2)} = \frac{(b-P_1)(a-P_0)}{(a-P_1)(b-P_0)}$$

$$(b-P_2)(b-P_0)(a-P_1)^2 = (a-P_0)(a-P_2)(b-P_1)^2$$

This is a quadratic equation of a, with solutions: $a_1 = b$ and $a_2 = \frac{b\left(P_0P_2-P_1{}^2\right)-P_1(P_1P_0-2P_0P_2+P_1P_2)}{b(P_0-2P_1+P_2)+P_1{}^2-P_0P_2}$. Discard a_1 which contradicts the known condition, so the expression for a is:

$$a = \frac{b\left(P_0P_2 - P_1{}^2\right) - P_1\left(P_1P_0 - 2P_0P_2 + P_1P_2\right)}{b\left(P_0 - 2P_1 + P_2\right) + P_1{}^2 - P_0P_2}$$

The expressions of b and P_0 can be derived similarly.

Practice problem set 6

Solve the following problems.

1. Suppose we have the following data for the earth's population:

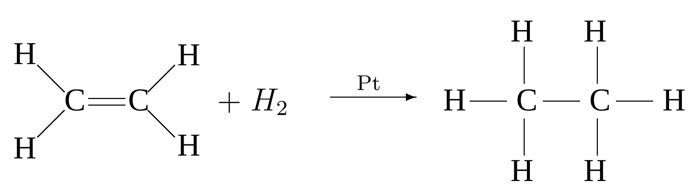
Year	1960	1970	1980
Population (in billions)	3.01	3.59	4.13

   Assume the population grows logistically. Estimate the earth's maximum capacity for human growth.

2. Because of limited food and space, a squirrel population can not exceed 1000. It growth at a rate proportional both to the existing population and to the attainable additional population. If there were 250 squirrels two years ago and the population is 530 now. Estimate the squirrel population four years ago.

3. Two chemicals A, B react together, one molecule of A combining with one of B. Initially their concentrations are equal and in two hours they are halved. When will they be one-quarter of their initial value?

4. In the hydrogenation of an alkene, a double bond between carbon atoms is converted to a single bond by the addition of hydrogen gass with the help of a platinum catalyst. In the simple example of ethene, the overall reaction is:



   Assuming that initially 0.10 M of ethene and 0.20 M of hydrogen gas are placed together under the proper conditions for the reaction to occur, the differential equation governing the reaction is

$$\frac{dx}{dt} = k(0.1-x)(0.2-x)$$

   where x denotes the concentration of the product (ethane, C_2H_6). Find the reaction constant k, if after 20 minutes the concentration of ethane is 0.060 M.

Chapter 7

Analysis of Logistic Equation*

Purpose: *To explore the various properties of the logistic equation. This chapter is not covered on the AP Calculus exams.*

From the previous studies, we have learned that the logistic equation is an S-shaped curve that can be used to model population growth and we have used it to solve some real world problems. In this chapter we are going to analyze the logistic equation to further understand its properties. Although this chapter is not a requirement for the AP Calculus exam, it is useful knowledge of the logistic equation and the method used in this chapter can be applied to the analysis of other functions.

Here, the logistic equation and its explicit solution are listed again for reference:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) \tag{5.2}$$

$$P = \frac{KP_0}{P_0 + (K - P_0)\, e^{-rt}} \tag{5.3}$$

In the following analysis, we will try to learn as much as possible about the logistic equation's properties directly from equation 5.2, and to use equation 5.3 only when necessary. By doing so, we can possibly delay or even save the extra effort of having to solve equation 5.2, although in this book we have already done so.

The logistic equation belongs to a class of differential equations, in which the independent variable, in this case t, does not appear explicitly. Such equations are called **autonomous** and have the form $\frac{dP}{dt} = f(P)$. For equation 5.2, $f(P) = rP\left(1 - \frac{P}{K}\right)$, which is a quadratic function of P. The following figure shows the curve of $f(P)$ under the condition $r > 0$.

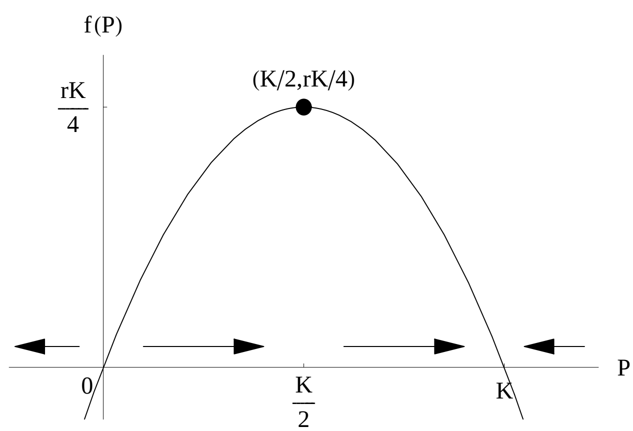


Figure 7.1: Quadratic function $f(P) = rP\left(1 - \frac{P}{K}\right)$ with $r > 0$

First we notice that the curve of $f(P)$ in Figure 7.1 has two P intercepts at $P = 0$ and $P = K$, respectively. These two points correspond to the P values that make $\frac{dP}{dt} = f(P) = 0$, which means no change in the P value with respect to t. Therefore $P = 0$ and $P = K$ are two constant solutions of the logistic equation.

Next we can see that when $r > 0$, the curve of $f(P)$ between $P = 0$ and $P = K$ is above the P axis, which means $\frac{dP}{dt} > 0$, or the solution $P(t)$ is increasing. In Figure 7.1 this is indicated by the two arrows pointing to the right, between $P = 0$ and $P = K$. If the initial condition P_0 satisfies $0 < P_0 < K$, then $P(t)$ will increase towards K, but never reach K.[1] So in this case the solution curve $P(t)$ is bounded between $P = 0$ and $P = K$. Also from Figure 7.1, the values of $f(P)$ are small near the curve's P intercepts compared to its values near the vertex of the parabola. This means that the solution curve $P(t)$ is relatively flat near $P = 0$ and $P = K$, and it becomes steeper near the vertex where $P = \frac{K}{2}$.

We can also determine the concavity of the $P(t)$ curve and the location of its point of inflection by finding $\frac{d^2P}{dt^2}$. From $\frac{dP}{dt} = f(P)$, we apply the chain rule to obtain:

$$\frac{d^2P}{dt^2} = f'(P)\frac{dP}{dt} = f'(P)f(P)$$

The above equation shows that when $f(P)$ and $f'(P)$ have the same sign, then $\frac{d^2P}{dt^2} > 0$; otherwise $\frac{d^2P}{dt^2} < 0$. From Figure 7.1, when $0 < P < \frac{K}{2}$, both $f(P)$ and $f'(P)$ are positive, so $\frac{d^2P}{dt^2}$ is positive, or the $P(t)$ curve is concave up. When $\frac{K}{2} < P < K$, $f(P)$ is positive, $f'(P)$ is negative, so the $P(t)$ curve is concave down. It can also be seen that $P(t)$ has a point of inflection at $P = \frac{K}{2}$ where $\frac{d^2P}{dt^2}$ changes sign and the $P(t)$ curve reaches its maximum slope of $\frac{rK}{4}$ at the point of inflection.

To find the value of t at the point of inflection, we can set $P = \frac{K}{2}$ in equation 5.3 and solve for t:

$$\begin{aligned}
\frac{K}{2} &= \frac{KP_0}{P_0 + (K - P_0)e^{-rt}} \\
2P_0 &= P_0 + (K - P_0)e^{-rt} \\
\frac{P_0}{K - P_0} &= e^{-rt} \\
-rt &= \ln\left(\frac{P_0}{K - P_0}\right)
\end{aligned}$$

$$t_i = \frac{1}{r}\ln\left(\frac{K}{P_0} - 1\right) \tag{7.1}$$

(Time t_i at the point of inflection)

So far we have analyzed the logistic equation under the condition $r > 0$ and $0 \le P_0 \ge K$. The $P(t)$ curves satisfying this condition are shown in Figure 7.2, between horizontal lines $P = 0$ and $P = K$. From the figure it can be seen that if $0 < P_0 < \frac{K}{2}$, the curve of $P(t)$

[1] The fundamental existence and uniqueness theorem states that particular solutions to a differential equation are all unique curves that never intersect. In our case, since $P = K$ is a solution, no other solution may intersect this line.

will intersect the horizontal line $P = \frac{K}{2}$, therefore passing through its point of inflection. When $\frac{K}{2} < P_0 < K$, the curve of $P(t)$ does not have a point of inflection for $t > 0$. Under the

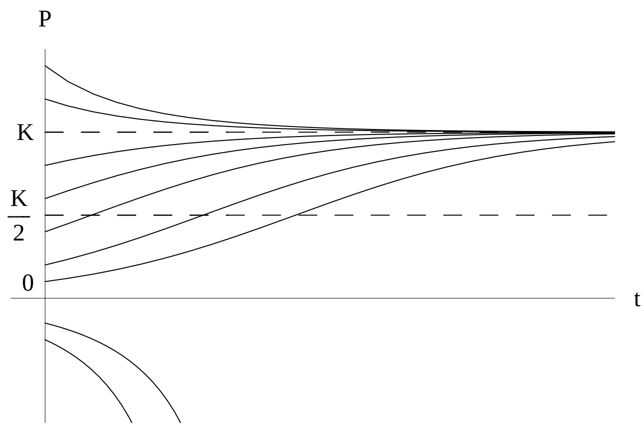


Figure 7.2: Logistic solution curve $P = \frac{KP_0}{P_0+(K-P_0)e^{-rt}}$ with $r > 0$

condition $r > 0$, with $P_0 > K$ or $P_0 < 0$, Figure 7.1 indicates that the $P(t)$ curve should be decreasing because the curve of $f(P)$ is below the P axis. When $P_0 > K$, $P(t)$ is concave up because the quantities $f(P)$ and $f'(P)$ are both negative. When $P_0 < 0$, $P(t)$ is concave down because $f(P)$ and $f'(P)$ have opposite signs. These curves are also shown in Figure 7.2.

It is worth mentioning that if $P_0 > K$ or $P_0 < 0$, the logistic solution will approach infinite at a certain finite time, or the $P(t)$ curve will have a vertical asymptote. This time t_∞ can be calculated by setting the denominator of equation 5.3 to 0, and solving for t:

$$P_0 + (K - P_0)\, e^{-rt} = 0$$

$$\ln\left(\frac{P_0 - K}{P_0}\right) = rt$$

$$t_\infty = \frac{1}{r}\ln\left(1 - \frac{K}{P_0}\right) \tag{7.2}$$

(Time point t_∞ at which function $P(t)$ approaches infinite)

In Figure 7.2 the asymptotes of the $P(t)$ curves lie outside the graph's domain and are not shown. When we plot values of t towards positive and negative infinite, these asymptotes will appear, as shown in Figure 7.4.

If $r < 0$ and $0 < P_0 < K$ we can apply a similar analysis using equations 5.2 and 5.3 to conclude that $P(t)$ is decreasing between 0 and K, and its curve is like a horizontally flipped 'S', as shown in Figure 7.3. If the initial condition $\frac{K}{2} < P_0 < K$, the curve of $P(t)$ will pass through its point of inflection at $P = \frac{K}{2}$. If $0 < P_0 < \frac{K}{2}$, the curve of $P(t)$ does not have a point of inflection for $t > 0$. If $P_0 > K$, the $P(t)$ is increasing and concave up; if $P_0 < 0$, the $P(t)$ curve is also increasing but concave down.

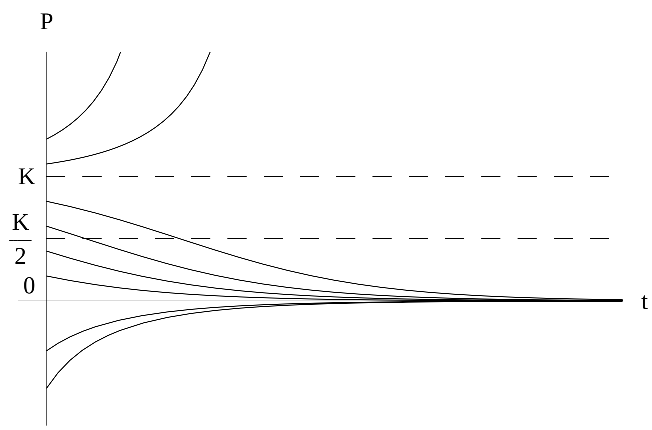


Figure 7.3: Logistic solution curve $P = \frac{KP_0}{P_0 + (K - P_0)e^{-rt}}$ with $r < 0$

Finally, we are going to show a "full picture" of the logistic solution curves by expanding the range of t values sufficiently large to include vertical asymptotes. Figure 7.4 shows the $P(t)$ curves under the condition $r > 0$. The three solid lines that intersect the P-axis are under the conditions $P_0 > K$, $0 < P_0 < K$, and $P_0 < 0$, respectively, which we are already familiar with. The arch at the lower left corner is the other branch of the $P_0 > K$ curve. In this case $t_\infty < 0$ because the argument of the logarithm in equation 7.2 is less than 1, so the logarithm is negative and $r > 0$. And the arch at the right upper corner is the other branch of the $P_0 < 0$ curve with $t_\infty > 0$ because in this case the logarithm in equation 7.2 is positive. Therefore only three logistic equation curves are plotted in Figure 7.4.

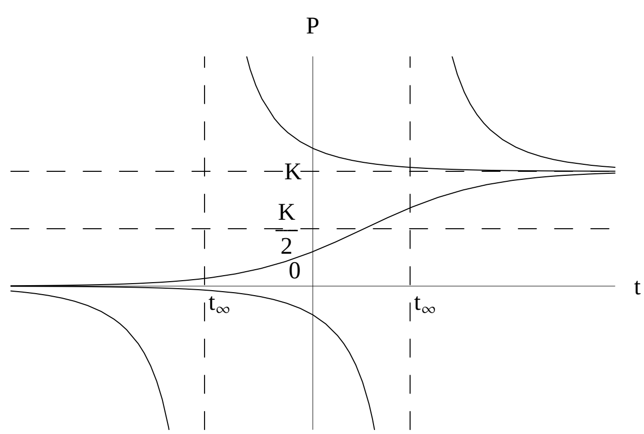


Figure 7.4: Logistic solution $P = \frac{KP_0}{P_0 + (K - P_0)e^{-rt}}$ with $r > 0$

Figure 7.5 shows the logistic solution curves under the condition $r < 0$. In this case the vertical asymptotes in the cases of $P_0 > K$ and $P_0 < 0$ switch to the other side of the P-axis (compare this with Figure 7.4). The analysis is very similar to the previous case.

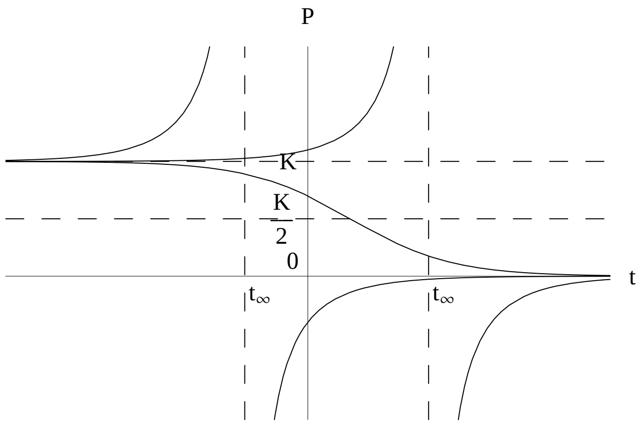


Figure 7.5: Logistic solution $P = \frac{KP_0}{P_0 + (K - P_0)e^{-rt}}$ with $r < 0$

Example 7.1

A contagious disease begins to spread in a community of 5000 people. Initially one person has it. After 20 days the spread of the disease appears to slow down. Assume the disease spreads according to the logistic model. Find how many people will be infected after 25 days.

Solution:

According to the given condition, we have $K = 5000$, $P_0 = 1$ and $r > 0$, so the function $P(t)$ is an S-shaped curve with a point of inflection at $\frac{K}{2} = 2500$. Since $\frac{dP}{dt}$ reaches its maximum value on the 20th day, it indicates that $t = 20$ is at the point of inflection, so we have $P(20) = \frac{K}{2} = 2500$. Substitute these values into equation 5.3:

$$2500 = \frac{5000}{1 + (5000 - 1)\, e^{-20r}}$$

$$e^{-r} = \left(\frac{1}{4999}\right)^{\frac{1}{20}}$$

After 25 days,

$$P = \frac{5000}{1 + (5000 - 1)\, e^{-25r}}$$

$$P = \frac{5000}{1 + (5000 - 1)\left(\frac{1}{4999}\right)^{\frac{25}{20}}} \approx 4469 \text{ people}$$

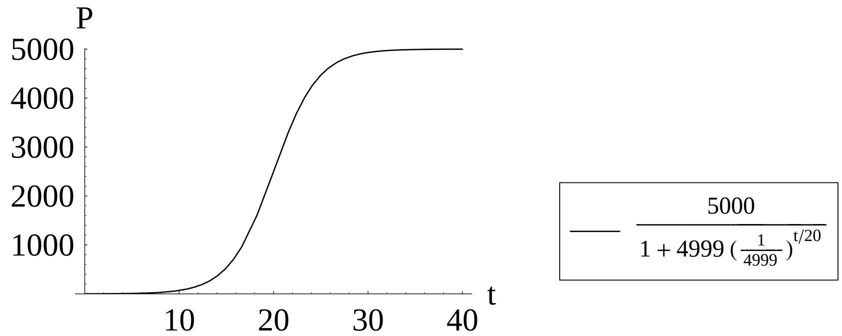


Figure 7.6: Graph for Example 7.1

Practice problem set 7

Solve the following problems:

1. A contagious disease begins to spread in a community of 2000 people. This disease spreads by contact and once a person has it, he/she will immediately and forever infect others. Initially one person has it, and the spread of the epidemic appears to lessen after two weeks. Find how many people have had the disease at any time t (in weeks).

2. An epidemic of a plague spreads through a town of 50000. Originally one person had it, and now after two weeks 20 people have it.

   a) When will the spread be most rapid?

   b) When will 2000 people have it?

3. In a second order chemical reaction $2A + 3B \to P$, 2 molecules of A combine with 3 molecules of B to form one molecule of P. Assume that the rate of the reaction is directly proportional to the concentrations of A and B, and initially chemical P does not exist. Let x be the concentration of P at time t, a and b be the initial concentrations of A and B, respectively. Express the reaction constant k in terms of x, t, a and b.

4. Apply the method in this chapter to analyze the second order growth equation. Use equation 6.7 as much as possible, and equation 6.8 only when necessary. Draw a figure similar to Figure 7.1 to help the analysis.

Chapter 8

The Hyperbolic Forms*

Purpose: *To explore the relationships between the logistic solution and hyperbolic functions. This chapter is not a requirement of the AP Calculus exam.*

Although the AP Calculus Exam does not cover hyperbolic functions, this chapter can help you better understand the nature of the logistic equation if you are familiar with the hyperbolic tangent ($\tanh$) and cotangent ($\coth$) functions. Figure 8.1 shows the two hyperbolic functions and their graphs.

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \text{ and } \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

These curves clearly bear a similarity to the shape of the logistic solution curves shown in

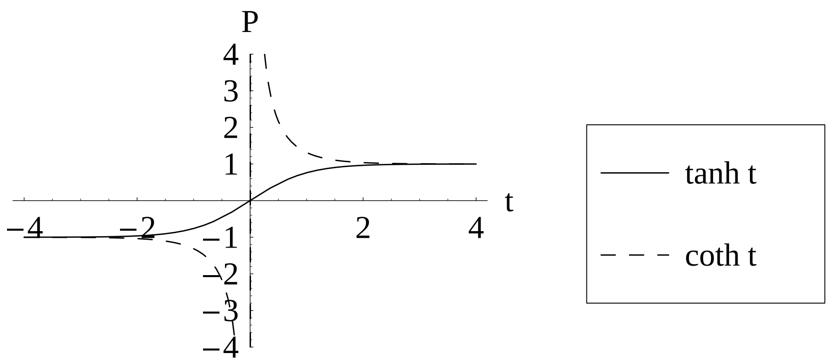


Figure 8.1: Hyperbolic tangent and cotangent curves.

the previous chapter. In fact, the logistic solution is simply a transformation of one of the two hyperbolic functions under different conditions.[1]

The $\tanh$ function resembles the logistic function when $0 < P_0 < K$. From the previous discussions we already know that under this condition the logistic equation is an S-shaped curve with a point of inflection at $(t_i, P_i) = \left(\frac{1}{r}\ln\left(\frac{K}{P_0} - 1\right), \frac{K}{2}\right)$, where P_i denotes the value of function P at the point of inflection. Let us define a new function $P_\tau(t)$, which has a curve identical to the logistic equation curve except that its point of inflection is at the origin. In other words, $P_\tau(t)$ is the logistic function $P(t)$ translated by $t = -\frac{1}{r}\ln\left(\frac{K}{P_0} - 1\right)$ and $P = -\frac{K}{2}$. So we have:

$$P_\tau(t) = P(t + t_i) - P_i \tag{8.1}$$

$$P(t) = P_\tau(t - t_i) + P_i \tag{8.2}$$

Apply equation 8.1 to equation 5.3:

$$P = \frac{KP_0}{P_0 + (K - P_0)e^{-rt}} = \frac{K}{1 + \left(\frac{K}{P_0} - 1\right)e^{-rt}}$$

[1]Bradley, D. M. "Verhulst's logistic curve," *The College Mathematics Journal*, 32 (2), 2001, pp. 94-98.

$$P_\tau = \frac{K}{1+\left(\frac{K}{P_0}-1\right)e^{-r(t+t_i)}} - \frac{K}{2}$$

$$P_\tau = \frac{K}{1+\left(\frac{K}{P_0}-1\right)e^{-r\left(t+\frac{1}{r}\ln\left(\frac{K}{P_0}-1\right)\right)}} - \frac{K}{2}$$

$$P_\tau = \frac{K}{1+\left(\frac{K}{P_0}-1\right)e^{-rt-\ln\left(\frac{K}{P_0}-1\right)}} - \frac{K}{2}$$

Since $xe^{-\ln x} = 1$, letting $x = \left(\frac{K}{P_0}-1\right)$ gives

$$P_\tau = \frac{K}{1+e^{-rt}} - \frac{K}{2}$$

$$P_\tau = \frac{K - Ke^{-rt}}{2\left(1+e^{-rt}\right)}$$

$$P_\tau = \frac{K}{2}\left(\frac{1-e^{-rt}}{1+e^{-rt}}\right)$$

Recall that $\tanh x = \frac{x^x - e^{-x}}{x^x + e^{-x}} = \frac{1-e^{-2x}}{1+e^{-2x}}$, so

$$P_\tau = \frac{K}{2}\tanh\left(\frac{1}{2}rt\right)$$

Apply equation 8.2 to the above equation:

$$P = \frac{K}{2}\tanh\left(\frac{1}{2}r\left(t-t_i\right)\right) + \frac{K}{2} \tag{8.3}$$

(Hyperbolic form of logistic equation when $0 < P_0 < K$)

Figure 8.2 shows the relationship between the $\tanh$ function and the logistic function under the conditions $r > 0$ and $0 < P_0 < K$. The dashed line is the logistic function curve and the solid line is the $\tanh$ function curve.

Similarly, the $\coth$ function resembles the logistic function $P(t)$ when $P_0 > K$ or $P_0 < 0$. Under these conditions we define:

$$P_\tau(t) = P\left(t+t_\infty\right) - \frac{K}{2} \tag{8.4}$$

$$P(t) = P_\tau\left(t-t_\infty\right) + \frac{K}{2} \tag{8.5}$$

Recall that t_∞ is the time when function P approaches infinite. Apply equation 8.4 to equation 5.3:

$$P_\tau = \frac{K}{1+\left(\frac{K}{P_0}-1\right)e^{-r(t+t_\infty)}} - \frac{K}{2}$$

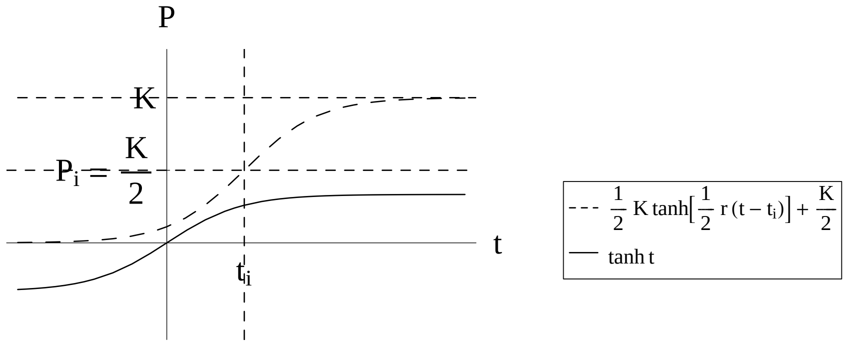


Figure 8.2: Relationship between tanh function and logistic function.

$$P_\tau = \frac{K}{1+\left(\frac{K}{P_0}-1\right)e^{-r\left(t+\frac{1}{r}\ln\left(1-\frac{K}{P_0}\right)\right)}} - \frac{K}{2}$$

$$P_\tau = \frac{K}{1-e^{-rt}} - \frac{K}{2}$$

$$P_\tau = \frac{K+Ke^{-rt}}{2\left(1-e^{-rt}\right)} = \frac{K}{2}\left(\frac{1+e^{-rt}}{1-e^{-rt}}\right)$$

Because $\coth x = \frac{e^x+e^{-x}}{e^x-e^{-x}} = \frac{1+e^{-2x}}{1-e^{-2x}}$, so

$$P_\tau = \frac{K}{2}\coth\left(\frac{1}{2}rt\right)$$

Apply equation 8.5 to the equation above to obtain:

$$P = \frac{K}{2}\coth\left(\frac{1}{2}r\left(t-t_\infty\right)\right) + \frac{K}{2} \tag{8.6}$$

(Hyperbolic form of logistic equation when $P_0 < 0$ or $P_0 > K$)

Figure 8.3 shows the relationship between the $\coth$ function and the logistic function under the condition $r > 0$ and $P_0 > K$. The dashed line is the logistic function curve and the solid line is the $\coth$ function curve.

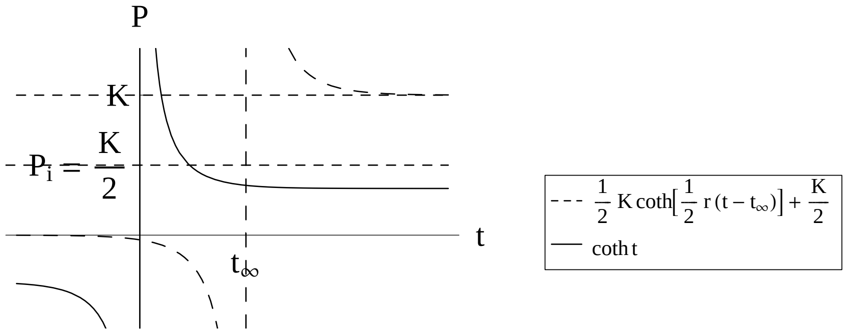


Figure 8.3: Relationship between coth function and logistic function.

Example 8.1
Express the logistic function of Example 5.1 in hyperbolic form.

Solution:
The logistic function is $P = \frac{1000}{1+19e^{-0.04t}}$ with $P_0 = 50$, $K = 1000$ and $r = 0.04$.
Since $0 < P_0 < K$,

$$t_i = \frac{1}{r} \ln\left(\frac{K}{P_0} - 1\right) = \frac{1}{0.04} \ln\left(\frac{1000}{50} - 1\right) = 25 \ln(19)$$

So the hyperbolic form of the solution is:

$$P(t) = 500 \left(\tanh\left(\frac{t - t_i}{50}\right) + 1\right) = 500 \left(\tanh\left(\frac{t - 25 \ln{(19)}}{50}\right) + 1\right)$$

Chapter 9

Slope Fields

Purpose: *To graphically express a differential equation using slope fields. This chapter is required by the AP Calculus BC exam and will be covered by the AB exam starting in 2004.*

Slope fields are a way to visualize a differential equation. It is simply a graph that shows the slopes at points on the coordinate plane for a differential equation. Below is an example:

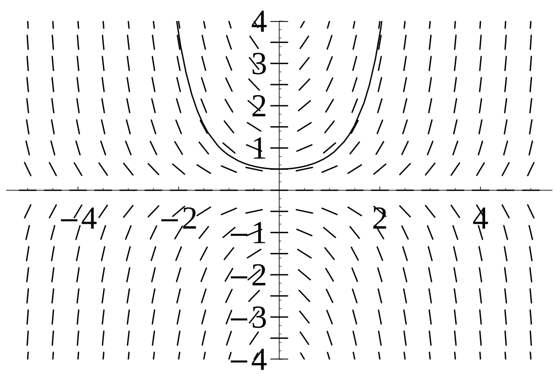


Figure 9.1: Slope field for $\frac{dy}{dx} = xy$

The specific solution, $y = \frac{1}{2}e^{\frac{x^2}{2}}$ for the initial condition (0,1), has been superimposed as the solid line on the field. Notice how the field's lines match up with the solution's curve. This is why slope fields are useful: they can show the shapes of the possible solutions (just follow the and connect the slope lines), as well as predict other values on the solution. Even though most graphing calculators can plot slope fields, you should know how to make them by hand. The AP test requires that you be able to identify what the slope field of a function looks like, without a graphing calculator. Making a slope field involves evaluating the differential equation at each point.

Example 9.1

Draw the slope field for the differential equation $\frac{dy}{dx} = xy^2$.

Solution:

Start off by making some observations about what the field should look like:

- On the axes, the slope is zero.
- Farther from the origin, the slopes are larger.
- In first and fourth quadrants, the slopes are positive.
- In the second and third quadrants, the slopes are negative.

Making observations like this will help you identify a slope field for a function.

Next, make a table of slope values, but for simpler functions like the one given, this can usually be done in your head. Only a few slope values are shown here:

x	y	slope
-2	-2	$(-2)(-2)^2 = -8$
-2	-1	$(-2)(-1)^2 = -2$
-2	1	$(-2)(1)^2 = -2$
-2	2	$(-2)(2)^2 = 8$
-1	-2	$(-1)(-2)^2 = -4$
-1	-1	$(-1)(-1)^2 = -1$
-1	1	$(-1)(1)^2 = -1$

Now, you are ready to graph the field. Draw a short line segment with the slope you calculated above for each of the points. The field should look something like this:

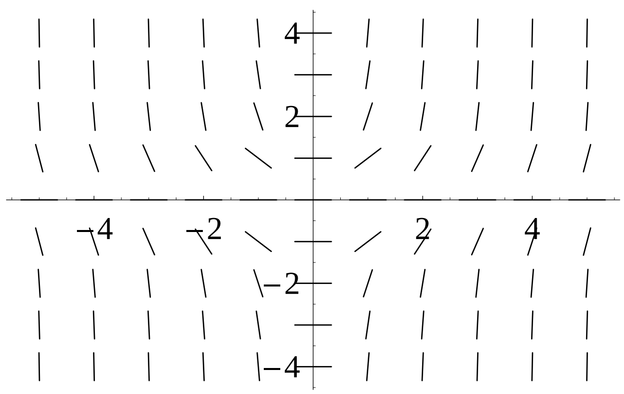
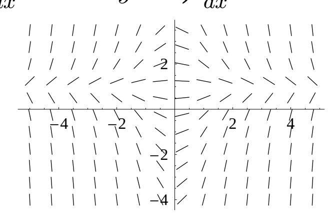


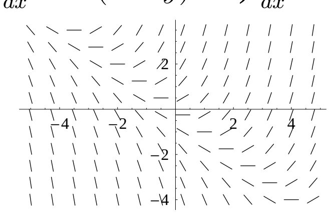
Figure 9.2: Slope field for $\frac{dy}{dx} = xy^2$

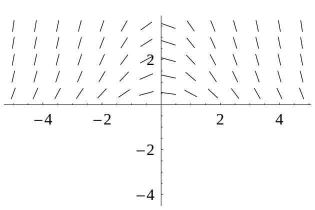
Example 9.2

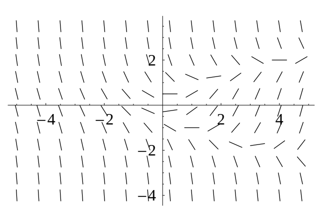
Match the following differential equations with their respective slope fields.

a) $\frac{dy}{dx} = x + y$ b) $\frac{dy}{dx} = x - y^2$ c) $\frac{dy}{dx} = x(1 - y)$ d) $\frac{dy}{dx} = -x\sqrt{y}$

1) 

2) 

3) 

4) 

Solution:

Since graph (3) is the only graph that does not exist below the x-axis, therefore, it must go with equation (d), which has the square root. Graph (1) appears to have zero slope at y = 1, and at x = 0. The only equation to meet these criteria is equation (c). Graph (2) appears to have zero slope along a diagonal $y = -x$. Equation (a) meets the criterion in this case. The only one left is graph (4) and equation (b), which makes sense because the graph seems to have zero slope along $y = \pm\sqrt{x}$. The answers are 1c, 2a, 3d, 4b.

Chapter 10

Euler's Method

Purpose: *To find numerical approximations to solutions of a differential equation. This chapter is required only by the AP Calculus BC exam.*

Not all differential equations can be solved explicitly to produce an exact solution. Euler's method is used to approximate a solution to a differential equation if an initial condition is known. It is based on increments and differentials; if the slope at one point is known, a neighboring point can be approximated by extending the slope line to a new x-coordinate:

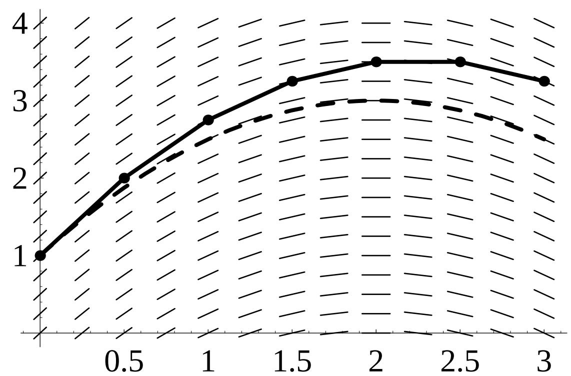


Figure 10.1: Comparison of Euler's method to exact solution.

The differential equation here is $\frac{dy}{dx} = 2 - x$, with the particular solution $\frac{1}{2}x^2 + 2x + 1$ graphed with a dotted line. To draw the approximation lines, start at $(0, 1)$ and draw a line segment with the slope at $(0, 1)$; it extends out to $(0.5, 2)$. This process is repeated, each time extending the line 0.5 units towards the right (and using the slope at the previously calculated point) until $(3, 3.25)$ is reached. That is the approximate solution of the differential equation, given the initial condition $(0, 1)$, and using a step size of 0.5. Below is the formal definition:

If $\frac{dy}{dx} = f(x, y)$, and you are given a point (x_1, y_1), then the next approximation is:

$$(x_1 + \Delta x, y_1 + \Delta x \cdot f(x_1, y_1)) \tag{10.1}$$

(Euler's method)

► NOTE

Δx (sometimes written h) is called the step size or increment; the smaller the step size, the more accurate the approximation. The step size can also be negative.

Example 10.1

Approximate the solution to the differential equation $\frac{dy}{dx} = y \ln x$ at $x = 2$ using Euler's method, given the initial condition $(1, 1)$ and using a step size of 0.25.

Solution:
It is often easiest to construct a table like this:

Old x	Old y	$f(x,y)$	New x	New y = Old $y + 0.25 \cdot f($Old x, Old $y)$
1	1	0	1.25	1
1.25	1	0.2231	1.5	1.0558
1.5	1.0558	0.4281	1.75	1.1628
1.75	1.1628	0.6507	2	1.3255

The solution is $(2, 1.3255)$. Seeing as how difficult the above problem was to do, even with only 4 iterations, methods for doing Euler's method on a calculator are provided later on in the chapter. On the AP test, you may not be able to use a calculator to do Euler's method problems.

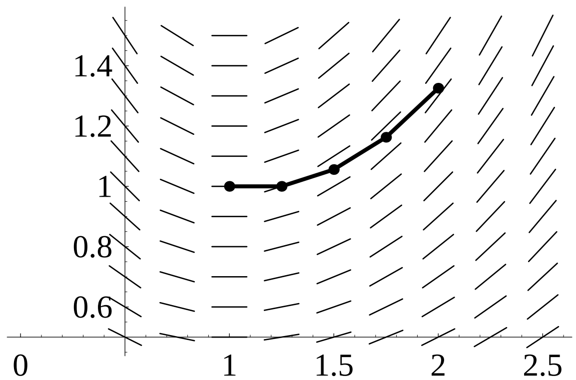


Figure 10.2: Euler's method approximation for $\frac{dy}{dx} = x \ln y$ from $x = 1$ to $x = 2$

Example 10.2
Find the difference between the value found with Euler's method and the actual value at $x = 0.5$ for the differential equation $\frac{dy}{dx} = -y \cos(8x)$ starting at $(0, 1)$ using a step size of 0.1.

Solution:

Old x	Old y	$f(x,y)$	New x	New y = Old $y + 0.25 \cdot f($Old x, Old $y)$
0	1	-1	0.1	0.9
0.1	0.9	-0.6270	0.2	0.8373
0.2	0.8373	0.0244	0.3	0.8397
0.3	0.8397	0.6192	0.4	0.9016
0.4	0.9016	0.9001	0.5	0.9916

The solution using Euler's method is $(0.5, 0.9916)$. To find the actual solution, we must solve the differential equation.

$$\frac{dy}{dx} = -y \cos(8x)$$
$$\int \frac{dy}{y} = -\int \cos(8x)\, dx$$
$$\ln|y| = -\frac{1}{8} \sin(8x) + C$$

$$y = e^{-\frac{1}{8}\sin(8x)+C}$$

Using the initial condition $C = 0$,

$$y = e^{-\frac{1}{8}\sin(8x)}$$

And the actual value is $e^{-\frac{1}{8}\sin(8(0.5))} \approx 1.0992$. The difference is $1.0992 - 0.9916 = 0.1076$.

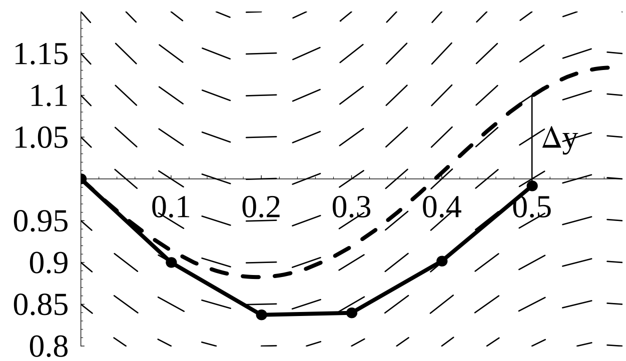


Figure 10.3: Euler's method approximation and exact solution for $\frac{dy}{dx} = -y\cos(8x)$

► **NOTE**

Euler's method will over underestimate if a curve is generally concave up (like above) and over estimate if a curve is generally concave down. For curves that change concavity, it is hard to tell whether the approximation is too large or small.

Using a calculator

It is a little known fact that the TI-86 and TI-89 have built in Euler's method capabilities when graphing differential equations. On the TI-86, set the graph mode to DifEq in the MODE menu. Now enter the differential equation (use t for x) and Q1 for y). You may actually graph more than one differential equation at a time, using Q2, Q3, etc. for the other equations, but the plot window would just get messy. Set the window and initial conditions in the WIND and INITC menus but be sure to set tStep to the step size, tMin to the beginning x value and tMax to the final x value. Press MORE to see the second set of menus and select FORMT. Here, select Euler instead of the default RK on the fifth line; this tells the calculator to use Euler's method when approximating the solutions. Now select GRAPH and the differential equation field and the approximate solution curve are plotted. Trace the curve (TRACE in the second set of menus) and the coordinates are shown at the bottom of the screen for each iteration of Euler's method.

On the TI-89, set the graph mode to 6:DIFF EQUATIONS. Enter the differential equation in the Y= editor (use t for x and y1 for y) as well as the initial y value in the line below it where it says yi1=. Again, you may graph multiple differential equations, but it is not recommended. In the Tools menu (F1), select 9:Format... and change Solution Method from RK to EULER. In the WINDOW editor set tstep to the step size, t0 to the beginning x value and tmax to the final x value (of course, also change the window bounds to fit the solution curve). Graph the equation and trace (F3 menu) the equation; at the bottom of the screen are xc and yc, the coordinates of each iteration.

The TI-82/83, and 83+ do not have built in Euler's method capabilities, however many programs already exist and are freely available online for downloading or typing into calculators. No program will be provided here because each type of calculator requires a different program.

Practice problem set 8

Solve these problems by hand and then try them using the calculator if possible.

1. Given that the solution to the differential equation $\frac{dy}{dx} = \frac{y}{x} + x$ passes through the point $(1, 2)$, approximate y when $x = 0.5$ using $h = -0.25$.
2. Using Euler's method with a step size of 1, estimate the population of a bacterial colony that follows the growth equation $\frac{dP}{dt} = 2.3P$ at $t = 3$ if currently it contains 100 bacteria.
3. To see how wrong Euler's method can be, find the error between the approximation and the actual value for the previous problem (include the sign).

Appendix A

Derivation of the Logistic Equation[1]

Motivated by Thomas Malthus' "An Essay on the Principle of Population" (1798), which predicted unlimited population growth, Pierre François Verhulst published his "logistique" equation (1838) to describe the inhibited growth of a population limited by a carrying capacity. Here we will discuss the logic behind the equation.

The differential form of the logistic equation, $\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$, is frequently used for modeling population growth. It is believed that a population, P, will grow at a rate which is somehow dependent upon its size, which would lead us to conclude the differential equation for exponential growth, $\frac{dP}{dt} = rP$, where r is the difference between the birth rate, b, and the death rate, d, and t is a unit of time. If r is positive, the population will unrealistically grow to an unlimited size. Populations do follow this type of growth patter for short periods of time, but this pattern is unrealistic for the long term. The factor $\left(1 - \frac{P}{K}\right)$ is added in the equation to produce inhibited growth.

Since the resources of a population are generally limited, it is reasonable to assume that as its density increases and it approaches the carrying capacity of its environment, the birth rate will decline and the death rate increase. Thus we could redefine the birth rate as $b = b_0 - k_b P$, where b_0 is the initial growth rate and k_b is the rate at which the birth rate declines as N grows. Similarly, the death rate could be redefined as $d = d_0 + k_d P$, where d_0 is the initial death rate and k_d is the rate at which the death rate increases as P grows. In other words, the birth and death rates can be interpreted as linearly related to the size of the population. P will stabilize, or reach a fixed point, if $b = d$, or $b_0 - k_b P = d_0 + k_d P$. Solving for P, we get a carrying capacity $K = \frac{b_0 - d_0}{k_b + k_d}$. Let $r = b_0 - d_0$. Then $k_b + k_d = \frac{r}{P}$. We will put this relationship aside for future reference.

By modify the differential equation for exponential grow, using the rate of growth in terms of our revised ideas about b and d,

$$\begin{aligned}
\frac{dP}{dt} &= rP = (b - d)\,P \\
&= \left[(b_0 - k_b P) - (d_0 + k_d P)\right] P \\
&= \left[(b_0 - d_0) - (k_b + k_d)\,P\right] P \\
&= \left[r - (k_b + k_d)\,P\right] P \quad \text{Let } (k_b + k_d) = \frac{r}{K} \quad \text{(from an earlier calculation)} \\
&= \left[r - r\frac{P}{K}\right] P = rP\left(1 - \frac{P}{K}\right)
\end{aligned}$$

[1]Wilson, E. O., and W. H. Bossert. 1971. A Primer of Population Biology. Sinauer Assoc., Inc., Sunderland, MA.

Free Response Problems

1. The radiation $R(t)$ in a substance decreases at a rate proportional to the amount present, or $\frac{dR}{dt} = kR$ where k is a constant and t is measured in years. The initial amount of radiation is 7200 rads. After three years the radiation has declined to 450 rads.
   (a) Express R as a function of t.
   (b) Find when the radiation will drop below 30 rads.
   (c) Find the half-life of this substance.

---

2. The growth of a bacterial colony in a petri dish is modeled by this differential equation:

$$\frac{dP}{dt} = 0.04P(100 - P)$$

   where $P \leq 100$ is measured in thousands of cells and $t \geq 0$ is measured in days.
   (a) Find the general solution to the differential equation.
   (b) If $P = 10$ when $t = 0$, find the particular solution to the differential equation.
   (c) Find how many people will eventually have the disease. Justify your answer.

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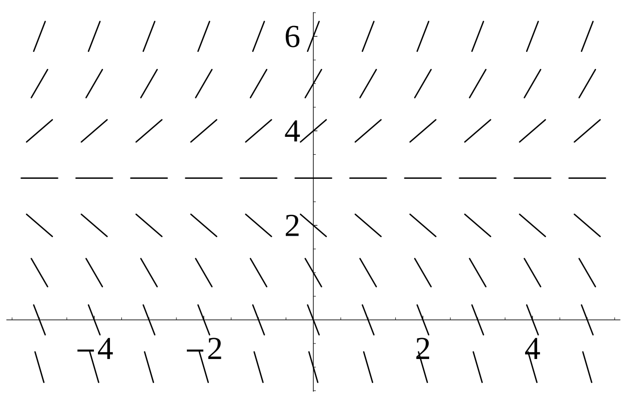
3. The slope field for the differential equation $\frac{dy}{dx} = y - 3$ (defined for all real numbers x) is shown below in the window $-5 < x < 5$ and $-1 < y < 6$.



   (a) Find the general solution to the differential equation in terms of an arbitrary constant C.
   (b) Find the particular solution that contains the point $(0, 1)$.
   (c) The graph above shows that some solutions approach $+\infty$ as $x \to \infty$ and others approach $-\infty$ as $x \to \infty$. Determine the C values that cause $\lim_{x\to\infty} y = +\infty t)$ and show your reasoning.

Answers to Practice Problems

Practice Problem Set 1 (Chapter 1)

1. $y = C\left(1 + x^2\right)$
2. $y^{\frac{3}{2}} = 9x^{\frac{1}{2}} + C$
3. $(x-1)^2 - (y+1)^2 + 2\ln\left|\frac{x+1}{y-1}\right| = C$
4. $\sin^2 y = C\frac{x-1}{x+1}$
5. $t^3y^2 = Ce^y$
6. $\sin x + y^2 = 1$
7. $y^2 + 2\ln|y| = x^2 - 4x + 5$
8. $2e^{x^2} + y^4 - 4y = 10$
9. $y = \ln\left|\frac{(x-3)^2(x+1)}{9}\right|$
10. If $y > a$ or $y < 0$, $y = \frac{Cae^{ax}}{Ce^{ax}-1}$;
    If $0 < y < a$, $y = \frac{Cae^{ax}}{Ce^{ax}+1}$

Practice Problem Set 2 (Chapter 2)

1. $y = x + Cxy$
2. $y = x - \frac{C}{x}$
3. $x^3 - 2y^3 = Cx$
4. $Cx = e^{\arcsin\frac{y}{x}}$
5. $(x+3y) - 9\ln|x+3y| = 7x + C$
6. $y = 3 + Ce^{-x^2}$
7. $y = 2(x-2)^3 + C(x-2)$
8. $y = 2e^{-x^2} + x^2 - 1$
9. $y\sin x + 5e^{\cos x} = 1$
10. $y = 2x^2\cos x + Cx\cos x$

Practice Problem Set 3 (Chapter 3)

1. $120\left(\frac{1}{2}\right)^{\frac{50}{74}} \approx 75$ grams
2. $3\ln\frac{25}{13}/\ln\frac{13}{6} \approx 2.54$ hours
3. $400(2)^5 = 12800$ cells
4. $\frac{10\ln 0.5}{\ln 0.32} \approx 6.08$ grams
5. $\ln 2/0.0525 \approx 13.2$ years
6. $100\left(\frac{1}{2}\right)^{\frac{3400}{5730}} \approx 66.3$ %
7. $20(0.7)^{\frac{20}{3}} \approx 1.86$ candelas
8. $\ln 2/0.05 \approx 13.9$ (In the 14th year)
9. $\frac{5\ln 2}{\ln 2.5} \approx 3.78$ (In the fourth year)
10. $2.5 = 5e^{-t/20}$, $t \approx 13.9$ minutes
11. 1.15 seconds
12. $I = 10e^{-0.4t}$, $I \approx 0.15A$

Practice Problem Set 4 (Chapter 4)

1. $\frac{12\ln\frac{1}{3}}{\ln\frac{13}{15}} \approx 92$ days
2. $50\left(1 - \left(\frac{3}{5}\right)^{\frac{30}{15}}\right) = 32$ words
3. $27 - 20\left(\frac{5}{6}\right)^{\frac{10}{5}} \approx 13$°C
4. $45 = K\left(1 - e^{-5r}\right)$,
    $80 = K\left(1 - e^{-10r}\right)$, 202.5 m/s
5. $\frac{5\ln 0.001}{\ln 12 - \ln 35} \approx 32.3$ minutes
6. $\frac{5\ln 0.75}{\ln 0.8} \approx 6.45$ minutes
7. $3\ln\left(\frac{98.6-65}{72-65}\right)/\ln\left(\frac{7}{15}\right) \approx -6.17$ hours
    Therefore, 6.17 hours ago from now is approximately 8:49 AM
8. $\frac{dP}{dt} = (0.097 - 0.047)P - 30000$
9. $v(t) = 4\left(1 - e^{-2.45t}\right)$ m/s
10. $Q(t) = 80 - 78e^{-\frac{t}{20}}$ lbs.
11. $143\ \mu$C
12. 21.2 seconds

Practice Problem Set 5 (Chapter 5)

1. $P(t) = \frac{10000}{1+9(1/6)^t}$; $P(2) = 8000$; $t \approx 0.45$ months

2. $\frac{1500}{1+299\left(\frac{23}{598}\right)^{\frac{3}{2}}} \approx 203$ students

3. $\frac{3000000}{100+(29900)\left(\frac{743}{2093}\right)^{\frac{14}{7}}} \approx 776$ people

4. $\frac{dP}{dt} = 0.04P - 0.0002P^2$ or $\frac{dP}{dt} = 0.04P\left(1 - \frac{P}{200}\right)$; $P(5) \approx 1.83$ million

5. $x = b + \frac{a-b}{1+e^{r(a-b)t}}$; $\lim_{t\to+\infty} x(t) = b$

6. a) 1.40 days; b) 3.34 days

7. $2 = \frac{(6.5)(1.5)}{1.5+(6.5-1.5)e^{-10r}}$; $e^{-r} = 0.675^{\frac{1}{10}}$
   $1.5 + (5)(0.99) = \frac{(6.5)(1.5)}{1.5+(6.5-1.5)0.675^{\frac{t}{10}}}$,
   $t \approx 154$ s

8. $\frac{dP}{dt} = r(1000 - 2P)$; $e^{-r} = \left(\frac{1}{2}\right)^{\frac{1}{10}}$,
   $t \approx 11.6$ years

Practice Problem Set 6 (Chapter 6)

1. ≈ 6.05 billion

2. ≈ 90 squirrels

3. $\frac{dx}{dt} = r(a-x)^2$;
   $r = 0.5$, in six hours

4. $\frac{0.2-0.06}{0.1-0.06} = \frac{0.2-0}{0.1-0}e^{(0.2-0.1)20k}$;
   $k \approx 0.28\frac{mol}{L\cdot min}$

Practice Problem Set 7 (Chapter 7)

1. $e^{-r} = \left(\frac{1}{1999}\right)^{1/2}$, $P(t) = \frac{2000}{1+1999\left(\frac{1}{1999}\right)^{\frac{t}{2}}}$

2. a) in 5.22 weeks; b) in 3.10 weeks

3. $\frac{1}{t(3a-2b)} \ln \frac{b(a-2x)}{a(b-3x)}$

4. Const. solutions at $P = a$ and $P = b$. If $P_0 < a$, func. increasing, concave down. If $a < P_0 < \frac{a+b}{2}$, func. decreasing, concave up. If $\frac{a+b}{2} < P_0 < b$, func. decreasing, concave down. If $P_0 > b$, func. increasing, concave up.

Practice Problem Set 8 (Chapter 10)

1. $\frac{31}{48} \approx 0.6458$

2. ≈ 3594 bacteria

3. $P = 100e^{2.3(3)} \approx 99227$ bacteria
   (Difference of $3594 - 99227 = -95633$)

Answers to Free Response Questions

Note: Actual free response problems on the AP Exam are worth 9 points each. The point-value breakdowns given here are only approximates of what they would really be on the exam.

1. The radiation $R(t)$ in a substance decreases at a rate proportional to the amount present, or $\frac{dR}{dt} = kR$ where k is a constant and t is measured in years. The initial amount of radiation is 7200 rads. After three years the radiation has declined to 450 rads.

(a) Express R as a function of t. $\frac{dR}{dt} = kR$ $\int \frac{dR}{R} = \int k dt$ $\ln\lvert R\rvert = kt + C$ $R = Ae^{kt}$ where $A = e^C$ $7200 = Ae^0$; $A = 7200$ $450 = 7200e^{3k}$ $k \approx -0.924$ $R = 7200e^{-0.924t}$	1 point for separating the variables. 1 point for antiderivatives. 1 point for the constant of integration. 1 point for obtaining the initial condition and the k value (must be to three decimal places) Total: 4 points
(b) Find when the radiation will drop below 30 rads. $30 = 7200e^{-0.924t}$ $\frac{1}{240} = e^{-0.924t}$ $t = \frac{\ln 240}{0.924} \approx 5.931$ years	2 points for the correct answer. -1 if the k value is not consistent with that in part a. -1 if the answer is not to three decimal places. Total: 2 points
(c) Find the half-life of this substance. $R = 7200e^{-0.924t}$ $3600 = 7200e^{-0.924t}$ or $\frac{1}{2} = e^{-0.924t}$ $t = \frac{\ln 2}{0.924} \approx 0.750$ years	1 point for setting up the equation (2nd line, either is correct). 2 points for the rest of the solution. -1 if the answer is not to three decimal places. Total: 3 points

2. The growth of a bacterial colony in a petri dish is modeled by this differential equation:

$$\frac{dP}{dt} = 0.04P\,(100 - P)$$

where $P \leq 100$ is measured in thousands of cells and $t \geq 0$ is measured in days.

(a) Find the general solution to the differential equation. $\frac{dP}{dt} = 0.04P\,(100 - P)$ $\frac{dP}{P(100-P)} = 0.04dt$ $\frac{A}{P} + \frac{B}{100-P} = \frac{1}{P(100-P)}$; $A = \frac{1}{100}$, $B = -\frac{1}{100}$ $\frac{1}{100}\ln\lvert P\rvert - \frac{1}{100}\ln\lvert 100-P\rvert = 0.04t + C$ $\ln\left\lvert\frac{100}{P} - 1\right\rvert = -4t - 100C$ $P = \frac{100}{1+e^{-100C}e^{-4t}}$ $P = \frac{100}{1+Ae^{-4t}}$, where $A = e^{-100C}$	1 point for separating the variables. 1 point for antiderivatives. 1 point for the partial fraction decomposition. 1 point for the constant of integration. 1 point for solving for P. Total: 5 points 2/5 if there is no constant of integration. 0/5 if there is no separation of variables.
(b) If $P = 10$ when $t = 0$, find the particular solution to the differential equation. $10 = \frac{100}{1+Ae^{0}}$; $A = 9$ so $P = \frac{100}{1+9e^{-4t}}$	2 points for the correct answer. Total: 2 points
(c) Find how many people will eventually have the disease. Justify your answer. The number of people that will eventually have the disease is the limit of P as t goes to infinite: $\lim_{t\to\infty} \frac{100}{1+9e^{-4t}} = 100$ people	1 point for the explanation 1 point for the limit Total: 2 points

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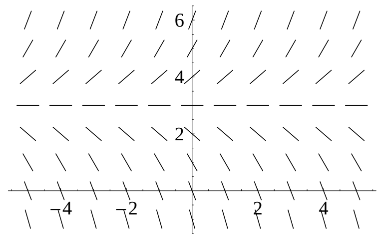

(a) Find the general solution to the differential equation in terms of an arbitrary constant C. $\frac{dy}{dx} = y - 3$ $\frac{dy}{y-3} = dx$ $\ln(y-3) = x + C$ $y = e^{x+C} + 3$ $y = e^C e^x + 3$	1 point for separating the variables. 1 point for antiderivatives. 1 point for the constant of integration. 1 point for solving for y.. Total: 4 points
(b) Find the particular solution that contains the point $(0, 1)$. $1 = e^C e^0 + 3$ $e^C = -2$ $y = -2e^x + 3$	Total: 2 points
(c) The graph above shows that some solutions approach $+\infty$ as $x \to \infty$ and others approach $-\infty$ as $x \to \infty$. Determine the C values that cause $\lim_{x\to\infty} y = +\infty$ and show your reasoning. In order for $\lim_{x\to\infty} e^C e^x + 3 = +\infty$, the coefficient e^C must be positive so then C can be any real number.	2 points for an explanation. 1 point for a correct condition on C. Total: 3 points

Editor's Notes

This is not an ordinary AP test preparation book. It substantially enhances the reader's knowledge of differential equations modeling real world growth processes. The book makes its unique contributions on the following topics:

1. It shows that four types of growth modeling differential equations, from the simple exponential growth and decay to the more complicated second order growth, can all be expressed in implicit, highly symmetric forms so that their common nature is revealed.
2. These implicit solution forms can be used to derive parameter expressions of the logistic and second order growth equations that are difficult to obtain using the more conventional explicit solutions.
3. The implicit solution forms in some cases provide a faster way to solve the more complicated logistic and second order growth problems. This book has included specific examples and practice problems to demonstrate this approach.
4. It derives many of the properties of the logistic solution from the original logistic equation, not from its solution function. Although this approach is not novel it certainly provides high school students an alternative method of analyzing an algebraic function similar to the logistic solution.
5. It proves by simple function curve translation that the logistic solution is just another form of the hyperbolic tangent and cotangent functions under different initial conditions.

This book was used at Olympia Institute as part of the test preparation curriculum for the AP Calculus and AP Physics Exams. It has been well received by the students. I recommend this book to students who are interested in gaining a more insightful knowledge of these growth modeling differential equations, especially the logistic equation. This book can also be useful to teachers, mathematicians, and engineers working in the related areas.

Professor Kuo Chen, Principal of Olympia Institute
950 Clement Street, San Francisco, CA 94118

About the Author

The author of this book, Victor Liu, has been a student at Olympia Institute since his sophomore year. Victor was born in Mountain View, California and has resided in the San Francisco Bay area all his life. He is a senior in the graduating class of 2003 at Monta Vista High School in Cupertino. He started taking the AP calculus and Honors physics courses in his sophomore year and received 5's on the AP Calculus BC and AB, Physics C (Mechanics and Electrical-Magnetic) exams in that year. In his junior year he received 5's on the AP Statistics, Computer Science, English Language and Chemistry exams. He maintains a 4.0 GPA in his sophomore and junior years. His interests include mathematics, physics, computer programming and networking, swimming, table tennis and music. He plans to study electrical engineering and computer science in college.